COMMUTATIVE ALGEBRA HW 2 SOLUTIONS

JC

Due in class Mon 5 September.

(1) A prime ideal J is *minimal* iff for all prime $I \subseteq J$ we have $I = J$. Show that every prime ideal contains a minimal prime ideal.

Note that any family of sets is partially ordered by $\supseteq$. So ZL easily implies that if a family of sets is such every chain has a lower bound, then every element contains a minimal element.

Now let P be the set of all prime ideals in R and let C be a chain in P. We may as well assume that C is non-empty. We need to check that C has a lower bound, so we try the most natural thing namely the intersection of C. As we mentioned in class it is trivial to check that $\bigcap C$ is an ideal, so we need to check it is prime. Let $x, y \notin \bigcap C$ and choose ideals J and K in C with $x \notin J$ and $y \notin K$. Since C is a chain without loss we have $J \subseteq K$, so that $y \notin J$. As J is prime $xy \notin J$ and so $xy \notin \bigcap C$.

(2) Identify the radical ideals of $\mathbb{Z}$.

A routine calculation shows that the radical ideals are (0) and (n) for $n > 0$ with no repeated prime factor (such n are sometimes called *quadratfrei*).

(3) Let R be a ring. The *power series ring* $R[[x]]$ consists of all expressions $\sum_{i=0}^{\infty} r_i x^i$ with the obvious definitions of $+$ and $\times$. Identify the units in $R[[x]]$.

Note that $(\sum_i a_i x^i)(\sum_j b_j x^j) = 1$ iff the system of equations $a_0 b_0 = 1$, $\sum_{j=0}^{n} a_i b_{n-i} = 0$ for all $n > 0$ is satisfied. So if $\sum_i a_i x^i$ is a unit then a_0 is a unit. Conversely if a_0 is a unit we may use the recursion $b_0 = a_0^{-1}$, $b_n = -a_0^{-1}(\sum_{i=1}^{n} a_i b_{n-1})$ to determine the coefficients of a multiplicative inverse for $\sum_i a_i x^i$.

Cultural remark: This is actually easier than the analysis of units in $R[x]$ for R not an ID.

(4) Let $\phi : R \to S$ be a ring HM and let J be an ideal of S. Show that if $I = \phi^{-1}[J]$ then I is an ideal of R, R/I is isomorphic to a subring of S/J, and "J is prime" implies "I is prime".

Consider the composition of ϕ and the usual quotient map $S \to S/J$. This has kernel I and its image is a subring of S/J, now appeal to the First Isomorphism Theorem for rings. If J is prime then S/J is an ID, subrings of IDs are always IDs so I is prime.

Cultural remark: this is another reason for focussing on prime ideals.