COMMUTATIVE ALGEBRA HW 21 SOLNS

JC

(1) Let R be a Noetherian local ring. Let M be the maxl ideal and let Q be an M-primary ideal Show that R/Q is Artinian.

Solution one (From Peter Lumsdaine, I like it better than my original solution): Since R/Q is Noetherian it is enough to show that R/Q has dimension zero. Any prime ideal of R/Q has form P/Q for P a prime ideal containing Q, but taking radicals we see that $P = \sqrt{P} \supseteq M = \sqrt{Q}$, so that $P = M$ and R/Q has only one prime ideal.

Solution two (My original idea) : Find n such that $M^n \subseteq Q$. Each quotient M^n/M^{n+1} has an R/M-module structure, and is N'ian. Since R/M is a field each M^n/M^{n+1} is FD, and hence Artinian. So R/M^n is Artinian and thus its homomorphic image R/Q is Artinian.

Solution two*: Find n with $M^n \subseteq Q$ then in R/Q we have that $(M/Q)^n = 0$. Appeal to result from book.

(2) Let $R_0 = k$ be a field, let R be the ring of polynomials $k[x_1, x_2]$ made into a graded ring in the obvious way. Let M_0 be a k-VS of dimension d and let $M = M_0[x_1, x_2]$ be the graded R-module of polynomials with coefficients from M_0.

Let $\lambda(M) = dim_k(M)$ for M in the set of FD k-VS's. and as in class define the Poincare series

$$P(M, t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n$$

Verify the predictions from the end of class:

$P(M, t)$ is of form $f/(1-t)^2$ for some $f \in \mathbb{Z}[t]$ and there is polynomial g such that $\lambda(M_n) = g(n)$ for all large n.

Let B be a k-basis of size d for M_0. Then M_n has a basis consisting of elements $bx_1^i x_2^{n-i}$ for $b \in B$ and $0 \leq i \leq n$, there are $d(n+1)$ elements in this basis. So

$$P(M, t) = \sum_{n=0}^{\infty} d(n+1)t^n = d/(1-t)^2$$