COMMUTATIVE ALGEBRA HW 13

JC

Due in class Fri 7 October.

(1) Let B be a ring. Show there exists a ring C with $B \leq C$ and an element $c \in C \setminus B$ such that c is integral over B.

Let $C = B[y]/(y^2)$ where we identify B with the set of cosets $b + (y^2)$ in C. Let $c = y + (y^2)$. Check it works.

Note: What is the point of this exercise? Just to emphasise the difference between the situation with fields: if B is an algebraically closed field and C is any extension *field* then all elements of C which are integral over B are actually in B.

(2) Let R be a ring, I an ideal of R and M an R-module. Show that M/IM and $R/I \otimes_R M$ are isomorphic as R/I-modules.

Low tech: verify that the map $(r + I, m) \mapsto rm + IM$ has the usual universal property. Suppose that ϕ is an R-bilinear map from $R/I \times M$ to some R-module C. Observe that

$$\phi(r + I, m) = \phi(1 + I, rm),$$

so the natural thing to try is to factor through a linear map $x + IM \mapsto \phi(1 + I, x)$, which clearly is linear and the unique thing that works but may not be well-defined. Then note that if $x_1 - x_2 \in IM$ then $x_1 - x_2 = \sum_j a_j m_j$ for some $a_j \in I$ and $m_j \in M$, and that

$$\phi(1 + I, \sum_j a_j m_j) = \sum_j (a_j + I, m_j) = \sum_j \phi(0, m_j) = 0.$$

So by bilinearity $\phi(1 + I, x_1) - \phi(1 + I, x_2) = 0$.

Hi tech: consider the exact sequence of R-modules $I \to R \to R/I \to 0$ with the usual maps, and tensor with M to get an exact (by a thm from class) $I \otimes M \to R \otimes M \to R/I \otimes M \to 0$. Using the standard IM between $R \otimes M$ and M we have a surjective map from M to $R/I \otimes M$ which maps m to $(1+I) \otimes M$.

Now the image of the HM (inclusion map tensor identity map) from $I \otimes M$ to $R \otimes M$ in the sequence above is all elements of form $\sum_j a_j \otimes m_j$. Under the IM between $R \otimes M$ and M this corresponds to IM, so by exactness the kernel of the HM from M to $R/I \otimes M$ given above is IM.

So we get an IM of R-modules in which $m + IM$ maps to $(1 + I) \otimes m$. Check it's an IM of R/I-modules (trivial!)

(3) Let A, B be rings with $A \leq B$ and suppose that $B \setminus A$ is closed under multiplication. Show that the integral closure of A in B (the set of elements of B which are integral over A) is precisely A.

Let $b \in B \setminus A$ and suppose for a contradiction that b is integral. Let $b^n = \sum_{i<n} a_i b^i$ be a relation of integral dependence with n chosen minimal. Note that $n > 1$ as $b \notin A$.

Now $b(b^{n-1} - \sum_{0<i<n} a_i b^{i-1}) = a_0$, but this is impossible as each of the factors on the LHS is in $B \setminus A$.