COMMUTATIVE ALGEBRA HW 10

JC

Due in class Fri 30 September.

(1) Read the handout on affine algebraic geometry.

(2) Let $k = \mathbb{C}$. Consider the variety $V(I)$ in $\mathbb{A}^3$ where $I = (x^2 - yz, xz - x)$. Show that V is the irredundant union of three irreducible varieties, and describe them by giving the prime ideal corresponding to each one.

It is best to think geometrically. $x(z-1) = 0$ holds iff either $x = 0$ or $z = 1$ (Seems reasonable: a single equation ought to define a set of dimension $3 - 1 = 2$, in this case a union of two planes. Later we expend a lot of effort on ideas of "dimension" in algebra)

If $x = 0$ then the equation $x^2 - yz = 0$ reduces to $yz = 0$, which of course is true iff $y = 0$ or $z = 0$. So this case gives us two lines contained in the final variety namely $x = y = 0$ and $x = z = 0$. These correspond to the ideals (x, y) and (x, z). Note that $k[x, y, z]/(x, y) \simeq k[z]$ which is an ID, and hence (x, y) is prime. A similar argument works for (x, z).

If $z = 1$ then we get the equation $y = x^2$, so now we are looking at the variety corresponding to the ideal $(z - 1, y - x^2)$. It is a general fact (Hint: use division) that for any ring ring R and any $r \in R$ we have $R[z]/(z - r) \simeq R$. So in particular $k[x, y, z]/(z - 1, y - x^2) \simeq k[x, y]/(y - x^2)$. Since $y - x^2$ is irreducible in the UFD $k[x, y]$ it is prime so that $k[x, y]/(y - x^2)$ is an ID. Thus $(z - 1, y - x^2)$ is prime in $k[x, y, z]$.

(3) (A small part of A and M 3.21)

A *homeomorphism* between two topological spaces X and Y is a bijection f between X and Y such that f and f^{-1} are both continuous (so f sets up a bijection between the open sets of X and of Y via the correspondence $O \mapsto f[O]$).

Let R be a ring, S an MC subset of R and $\phi : R \to S^{-1}R$ the map $\phi : r \mapsto r/1$. Let $X = Spec(R)$ and $Y = Spec(S^{-1}R)$ so that as we saw in a previous HW ϕ induces a continuous map $Spec(\phi)$ from Y to X. Let Z be the image of $Spec(\phi)$, and give Z the subspace topology. Show that $Spec(\phi)$ is a homeomorphism from Y to Z.

We recall that by results from class Y is in 1-1 correspondence with the set of primes $P \in X$ such that $P \cap S = \emptyset$ where P^e corresponds to P. When we apply the map $Spec(\phi)$ to P^e we get $P^{ec} = P$, so that Z is the set of primes disjoint from S and $Spec(\phi)$ gives a bijection from Y to Z in which P^e maps to P

We know that $Spec(\phi)$ is continuous. For every open set O of X the inverse image of $O \cap Z$ under $Spec(\phi)$ equals the inverse image of O under $Spec(\phi)$ which is open in Y. For the other direction we recall that the open sets in the spectrum $Spec(A)$ are the unions of "basic" open sets $O^A_r = \{P \in Spec(A) : r \notin A\}$, and so it will be enough to show that the image of $O^{S^{-1}R}_{a/s}$ under $Spec(\phi)$ is open in Z. But this is easy, since s is a unit

$$a/s \in P^e \iff a/1 \in P^e \iff a \in P^{ec} = P,$$

so the image is precisely $O^R_a \cap Z$.