COMMUTATIVE ALGEBRA HANDOUT 3: FUNCTORS, NATURALITY, ADJOINTS AND LIMITS

JC

1. Functors

The notion of morphism axiomatises the notion of "structure preserving map between mathematical objects". Now we look at structure preserving maps between categories. If $\mathcal{C}$ and $\mathcal{D}$ are categories then a (covariant) *functor* from $\mathcal{C}$ to $\mathcal{D}$ is a function from $Ob(\mathcal{C}) \cup Hom(\mathcal{C})$ to $Ob(\mathcal{D}) \cup Hom(\mathcal{D})$ such that

(1) For all $a \in Ob(\mathcal{C})$, $F(a) \in Ob(\mathcal{D})$.
(2) For all $f : a \to b$ in $Hom(\mathcal{C})$, $F(f) : F(a) \to F(b)$ in $Hom(\mathcal{D})$.
(3) For all $a \in Ob(\mathcal{C})$, $F(id_a) = id_{F(a)}$.
(4) For all $f : a \to b$ and $g : b \to c$ in $Hom(\mathcal{C})$, $F(g \circ f) = F(g) \circ F(f)$.

This is intuitively the right thing. Categories are defined by the distinction between objects and morphisms and the operations of *dom*, *cod*, *id* and $\circ$; functors just preserve all of this.

Example: consider the map U defined as follows. For every ring R, $U(R)$ is the group of units of R (to be completely explicit, the underlying set of $U(R)$ is the set of invertible elements in R and the group operation is the restriction of $\times_R$ to this set) For every ring HM $f : R \to S$, $U(f)$ is the restriction of f to the set of invertible elements in R. It is routine to check that this is a function from the category **Rings** of rings and ring HMs to the category **Groups** of groups and group HMs.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is just a covariant functor from $\mathcal{C}^{op}$ to $\mathcal{D}$. Explicitly it is a map F such that

(1) For all $a \in Ob(\mathcal{C})$, $F(a) \in Ob(\mathcal{D})$.
(2) For all $f : a \to b$ in $Hom(\mathcal{C})$, $F(f) : F(b) \to F(a)$.
(3) For all $a \in Ob(\mathcal{C})$, $F(id_a) = id_{F(a)}$.
(4) For all $f : a \to b$ and $g : b \to c$ in $Hom(\mathcal{C})$, $F(g \circ f) = F(f) \circ F(g)$.

Example: (from a recent HW) the functor *Spec* is a contravariant functor from **Rings** to the category **Top**

Example: let $\mathcal{C}$ be any category and let **Sets** be the category of sets and functions. For any object c we define a covariant functor $Hom_{\mathcal{C}}(c, -)$ from $\mathcal{C}$ to sets as follows: $Hom(c, d)$ is just (as usual) the set of morphisms $f : c \to d$ in the category $\mathcal{C}$ and for $f : d_1 \to d_2$, $Hom(c, f)$ is the map $g \mapsto f \circ g$ from $Hom(c, d_1)$ to $Hom(c, d_2)$. A contravariant functor $Hom(-, d)$ is defined similarly.

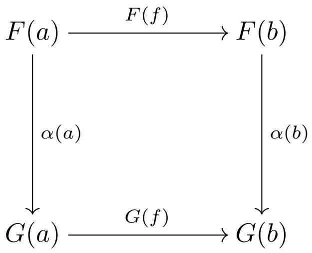
2. Naturality

Now we consider what should be a structure preserving transformation between functors (you knew this was coming). Let F and G be covariant functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* α from F to G is a map which assigns to each object

a of $\mathcal{C}$ a morphism $\alpha(a) : F(a) \to G(a)$ in $\mathcal{D}$, such that for all morphisms $f : a \to b$ in $\mathcal{C}$ we have the "naturality condition"

$$\alpha(b) \circ F(f) = G(f) \circ \alpha(a).$$

This is best pictured by the diagram



Example: Recall that if F is a field and V is an F-VS then the dual space V^* is the space of all linear maps from V to F. Given a map $\phi : V \to W$ we define $\phi^* : W^* \to V^*$ given by $\phi^* : f \mapsto f \circ \phi$. It is routine to check that this gives a contravariant functor from the category $F - \mathbf{VS}$ of F-VSs and linear maps to itself. Squaring we get a covariant functor $**$ which maps V to V^{**} and ϕ to ϕ^{**}. We will now define a natural transformation from the identity functor to the $**$-functor: for every vs V, $\alpha(V)$ is the map from $V = id(V)$ to V^{**} which takes $v \in V$ to the "evaluation map" $f \mapsto f(v)$ from V^* to F.

Cultural remark: typically transformations defined without making arbitrary choices (here the choice of a basis) will tend to be natural.

Cultural remark: given categories $\mathcal{C}$ and $\mathcal{D}$ there is a "functor category" whose objects are covariant functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations.

3. Product categories

The product category $\mathcal{C} \times \mathcal{D}$ is defined in the obvious way. Objects are pairs (c, d) where c is an object of $\mathcal{C}$ and d is an object of $\mathcal{D}$. A morphism is $(f_1, f_2) : (c_1, d_1) \to (c_2, d_2)$ where $f_1 : c_1 \to c_2$ and $f_2 : c_2 \to d_2$.

Example: we can bundle up the $Hom(c, -)$ and $Hom(-, d)$ functors into a single functor $Hom(-, -)$ from $\mathcal{C}^{op} \times \mathcal{C}$ to **Sets**. The point is that given morphisms $f : c_2 \to c_1$ and $g : d_1 \to d_2$ in $\mathcal{C}$ we may define $Hom(f, g)$ to be the function from $Hom(c_1, d_1)$ to $Hom(c_2, d_2)$ given by $h \mapsto g \circ h \circ f$. Think of this as "a functor of two variables which is contravariant in its first argument and covariant in its second".

4. Adjoint functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A common situation in mathematics is to have covariant functors $S : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{C}$ which are 'adjoints" of each other. We give the definition and some examples.

An *adjunction* from S to T is a family of maps $\nu_{c,d}$ where $c \in Ob(\mathcal{C})$ and $d \in OB(\mathcal{D})$, such that

(1) For all c and d, $\nu_{c,d}$ is a bijection between the sets $Hom_{\mathcal{D}}(S(c), d)$ and $Hom_{\mathcal{C}}(c, T(d))$.
(2) The $\nu_{c,d}$ are natural.

If such an adjunction exists we say that F is a *left adjoint* of G, and G is a *right adjoint* of F.

The naturality takes a bit of explaining. It is routine to see that we can think of $Hom_{\mathcal{D}}(T(-), -)$ and $Hom_{\mathcal{C}}(-, S(-))$ as both being functors from $\mathcal{C}^{op} \times \mathcal{D}$ to **Sets**. We want the $\nu_{C,D}$ to be the components of a natural transformation between them.

More explicitly we want that for all $f : c_2 \to c_1$ in $\mathcal{C}$ and all $g : d_1 \to d_2$ in $\mathcal{D}$ the diagram

$$\begin{array}{ccc} Hom(S(c_1), d_1) & \xrightarrow{Hom(S(f),g)} & Hom(S(c_2), d_2) \\ \downarrow{\scriptstyle \nu_{c_1,d_1}} & & \downarrow{\scriptstyle \nu_{c_2,d_2}} \\ Hom(c_1, T(d_1)) & \xrightarrow{Hom(f,T(g))} & Hom(c_2, T(d_2)) \end{array}$$

commutes.

Even more explicitly this means that for all $h : S(c_1) \to d_1$

$$\nu_{c_2,d_2}(g \circ h \circ S(f)) = T(g) \circ \nu_{c_1,d_1}(h) \circ f.$$

Example: Recall that if X is a set then the free abelian group $Fr(X)$ on X is the set of functions $f : X \to \mathbb{Z}$ which are zero except on some finite set, with the group operation being pointwise addition. We usually identify $x \in X$ with the function which is 1 at x and 0 elsewhere, so we can write a typical element of $Fr(X)$ as a finite sum $\sum_{i=1}^{k} n_i x_i$ for $n_i \in \mathbb{Z}$, $x_i \in X$.

We make Fr into a functor from **Sets** to **AbGroups** as follows: if X and Y are sets and $f : X \to Y$ is a function then $Fr(f) : \sum_i n_i x_i \mapsto \sum_i n_i f(x_i)$. To go the other way we define a "forgetful functor" Un from **AbGroups** to **Sets** as follows: $Un(G)$ is the underlying set of the abelian group G and $Un(\phi) = \phi$ for $\phi : G \to H$.

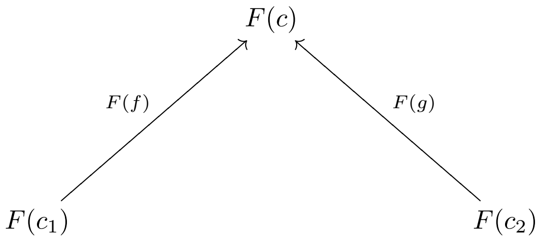
We claim that there is an adjunction from Fr to Un in which each function $f : X \to Un(G)$ corresponds to a HM from $Fr(X)$ to G given by the equation $\sum_i n_i x_i \mapsto \sum_i n_i f(x_i)$.

5. Limits

To illustrate the usefulness of adjointness we prove that if $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint then it preserves coproducts. More explicitly if

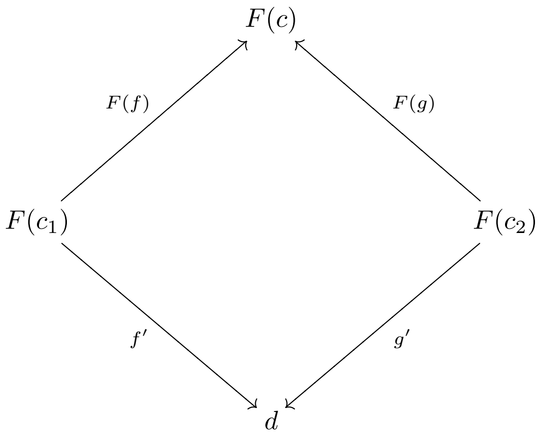
$$\begin{array}{ccccc} & & c & & \\ & \overset{f}{\nearrow} & & \overset{g}{\nwarrow} & \\ c_1 & & & & c_2 \end{array}$$

is a coproduct of c_1 and c_2 in $\mathcal{C}$ then

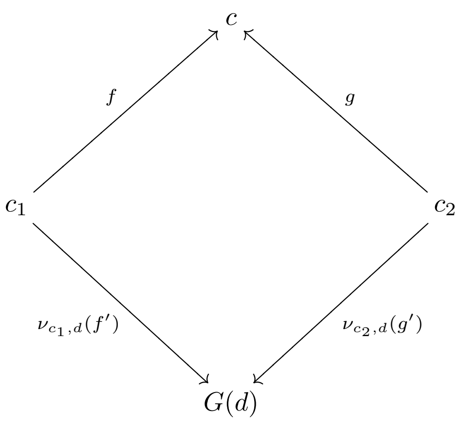


is a coproduct of $F(c_1)$ and $F(c_2)$ in $\mathcal{D}$.

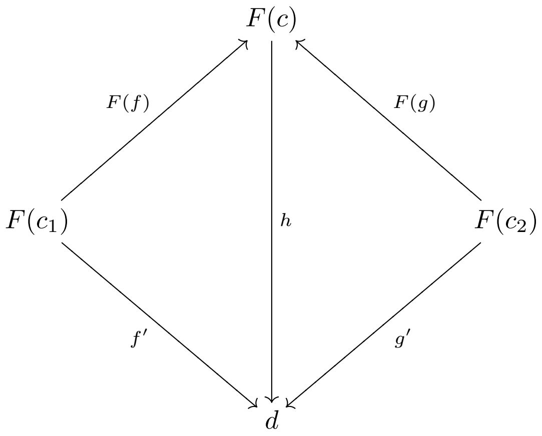
To see this suppose that we have



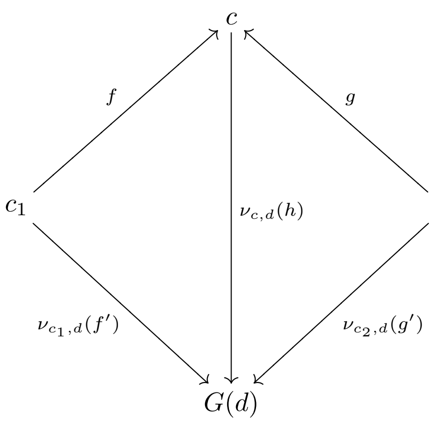
Back in $\mathcal{C}$ we have the corresponding



Now every morphism $c \to G(d)$ is of the form $\nu_{c,d}(h)$ for a unique $h : F(c) \to d$. By naturality $\nu_{c_1,d}(h \circ F(f)) = \nu_{c,d}(h) \circ f$ and $\nu_{c_2,d}(h \circ F(g)) = \nu_{c,d}(h) \circ g$, so that



commutes iff



There's a unique morphism making the bottom diagram commute so there's a unique morphism making the top diagram commute, and so F preserves coproducts as claimed.