T-79.7003, Lecture 6
Properties and stochastic models of real-world networks

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November 1st, 2013

Properties of real-world networks

Properties of real-world networks

diverse collections of graphs arising from different phenomena

are there typical patterns?

- static networks
  1. heavy tails
  2. clustering coefficients
  3. communities
  4. small diameters
- time-evolving networks
  1. densification
  2. shrinking diameters
- web graph
  1. bow-tie structure
  2. bipartite cliques
  3. compressibility

Heavy tails

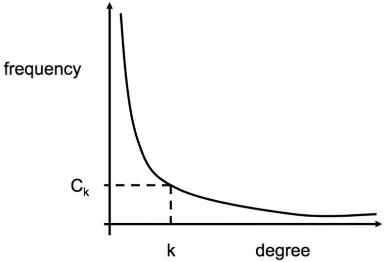
What do the proteins in our bodies, the Internet, a cool collection of atoms and sexual networks have in common? One man thinks he has the answer and it is going to transform the way we view the world.

Scientist 2002

Albert-László Barabási

Degree distribution

- C_k = number of vertices with degree k



- problem : find the probability distribution that fits best the observed data

Power-law degree distribution

- C_k = number of vertices with degree k, then

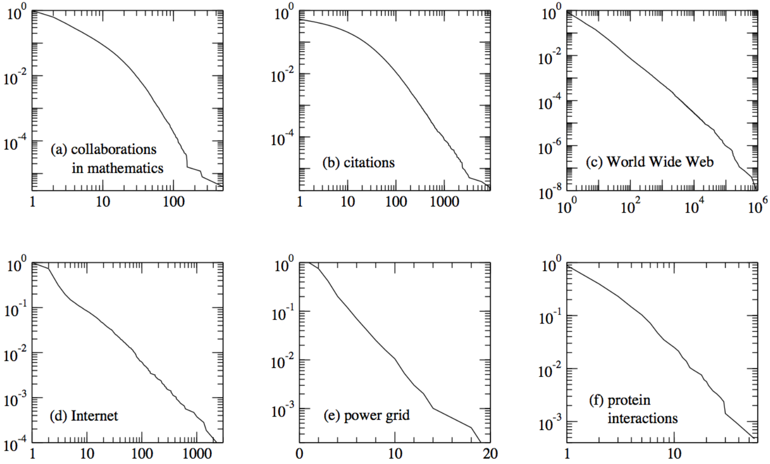
$$C_k = ck^{-\gamma}$$

with $\gamma > 1$, or

$$\ln C_k = \ln c - \gamma \ln k$$

- plotting $\ln C_k$ versus $\ln k$ gives a straight line with slope $-\gamma$
- heavy-tail distribution : there is a non-negligible fraction of nodes that has very high degree (hubs)
- scale free : average is not informative

Power-law degree distribution



power-laws in a wide variety of networks ([Newman, 2003])
sheer contrast with Erdős-Rényi random graphs

Power-law degree distribution

do the degrees follow a power-law distribution?

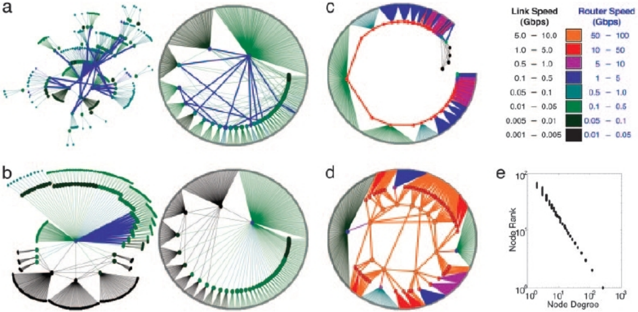
three problems with the initial studies

- graphs generated with traceroute sampling, which produces power-law distributions, even for regular graphs [Lakhina et al., 2003].
- methodological flaws in determining the exponent see [Clauset et al., 2009] for a proper methodology
- other distributions could potentially fit the data better but were not considered, e.g., lognormal.

disclaimer: we will be referring to these distributions as heavy-tailed, avoiding a specific characterization

Power-law degree distribution

- frequently, we hear about "scale-free networks" correct term is networks with scale-free degree distribution



all networks above have the same degree sequence but structurally are very different (source [Li et al., 2005])

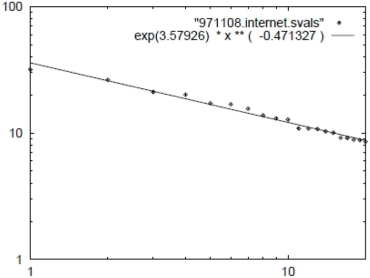
Maximum degree

- for random graphs, the maximum degree is highly concentrated around the average degree z
- for power-law graphs

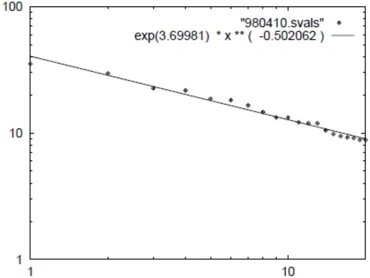
$$d_{\max} \approx n^{1/(\alpha-1)}$$

- hand-waving argument: solve $n \Pr[X \geq d] = \Theta(1)$

Heavy tails, eigenvalues



(a) Int-11-97

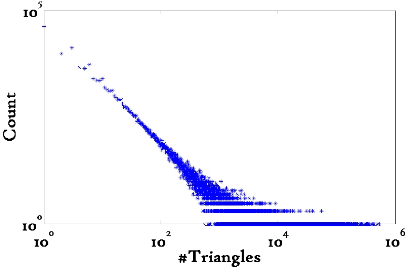


(b) Int-04-98

log-log plot of eigenvalues of the Internet graph in decreasing order

again a power law emerges [Faloutsos et al., 1999]

Heavy tails, triangles



- triangle distribution in `flickr`
- figure shows the count of nodes with *k* triangles vs. *k* in log-log scale
- again, heavy tails emerge [Tsourakakis, 2008]

Clustering coefficients

- a proposed measure to capture local clustering is the graph transitivity

$$T(G) = \frac{3 \times \text{number of triangles in the network}}{\text{number of connected triples of vertices}}$$

- captures "transitivity of clustering"
- if u is connected to v and
  v is connected to w, it is also likely that
  u is connected to w

Clustering coefficients

- alternative definition
- local clustering coefficient

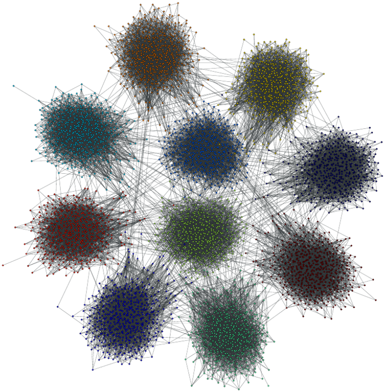
$$C_i = \frac{\text{Number of triangles connected to vertex } i}{\text{Number of triples centered at vertex } i}$$

- global clustering coefficient

$$C(G) = \frac{1}{n} \sum_i C_i$$

Community structure

loose definition of community: a set of vertices densely connected to each other and sparsely connected to the rest of the graph



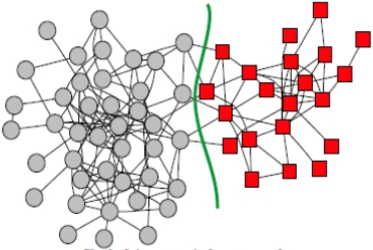
artificial communities:
http://projects.skewed.de/graph-tool/

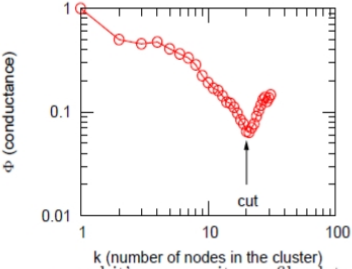
Community structure

[Leskovec et al., 2009]

- study community structure in an extensive collection of real-world networks
- authors introduce the network community profile plot
- it characterizes the best possible community over a range of scales

Community structure

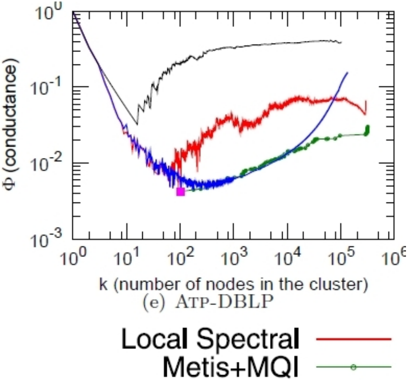




dolphins network and its NCP
(source [Leskovec et al., 2009])

Community structure

- do large-scale real-world networks have this nice artifical structure? NO!



NCP of a DBLP graph (source [Leskovec et al., 2009])

Community structure

important findings of [Leskovec et al., 2009]

1. up to a certain size k ($k \sim 100$ vertices) there exist good cuts
   - as the size increases so does the quality of the community
2. at the size k we observe the best possible community
   - such communities are typically connected to the remainder with a single edge
3. above the size k the community quality decreases
   - this is because they blend in and gradually disappear

Small-world phenomena

small worlds : graphs with short paths

- Stanley Milgram (1933-1984)
  "The man who shocked the world"
- obedience to authority (1963)
- small-World experiment (1967)

- we live in a small-world
- for criticism on the small-world experiment, see *"Could It Be a Big World After All? What the Milgram Papers in the Yale Archives Reveal About the Original Small World Study"* by Judith Kleinfeld

Small-world experiments

- letters were handed out to people in Nebraska to be sent to a target in Boston
- people were instructed to pass on the letters to someone they knew on first-name basis
- the letters that reached the destination (64 / 296) followed paths of length around 6
- *Six degrees of separation* : (play of John Guare)
- also:
  - the Kevin Bacon game
  - the Erdős number
- small-World project: http://smallworld.columbia.edu/index.html

Small diameter

proposed measures

- diameter : largest shortest-path over all pairs.
- effective diameter : upper bound of the shortest path of 90% of the pairs of vertices.
- average shortest path : average of the shortest paths over all pairs of vertices.
- characteristic path length : median of the shortest paths over all pairs of vertices.
- hop-plots : plot of $|N_h(u)|$, the number of neighbors of u at distance at most h, as a function of h [Faloutsos et al., 1999].

Time-evolving networks

J. Leskovec

J. Kleinberg

C. Faloutsos

[Leskovec et al., 2005]

- densification power law:

$$|E_t| \propto |V_t|^{\alpha} \qquad 1 \leq \alpha \leq 2$$

- shrinking diameters: diameter is shrinking over time.

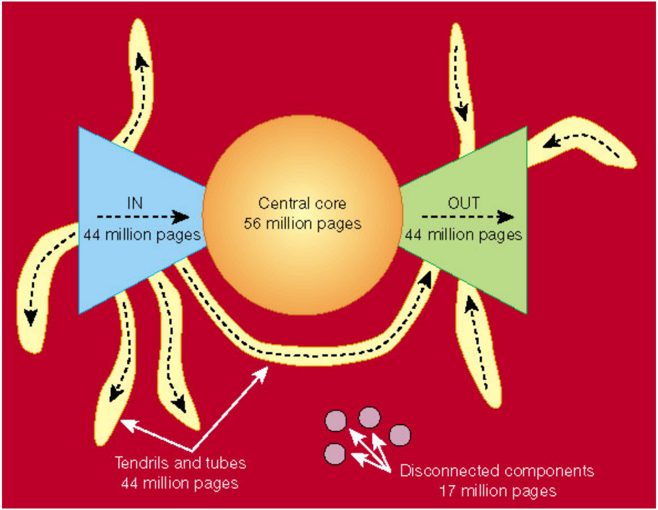
Web graph

- the Web graph is a particularly important real-world network

> *Few events in the history of computing have wrought as profound an influence on society as the advent and growth of the World Wide Web*
>
> [Kleinberg et al., 1999a]

- vertices correspond to static web pages
- directed edge (i, j) models a link from page i to page j
- will discuss two structural properties of the web graph:
  1. the bow-tie structure [Broder et al., 2000]
  2. abundance of bipartite cliques [Kleinberg et al., 1999a, Kumar et al., 2000]

Web is a bow-tie

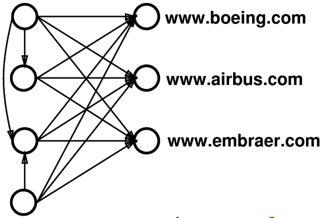


(source [Broder et al., 2000])

Bipartite subgraphs

- websites that are part of the same community frequently do not reference one another

  (competitive reasons, disagreements, ignorance) [Kumar et al., 1999].
- similar websites are *co-cited*
- therefore, web communities are characterized by dense directed bipartite subgraphs



(source [Kleinberg et al., 1999a])

Compressibility

In general, a graph can be stored by using $O(\log n)$ bits for edges. This is an upper bound. But what about lower bounds? But can be do better?

- Erdös-Rényi graphs require $\Omega(\log n)$ bits for each edge.
- Boldi and Vigna in a series of papers [Boldi and Vigna, 2004] demonstrate empirically that the Web-graph requires significantly smaller amount of bits per edge. Empirical evidence suggests $O(1)$ bits suffices.
- Work by Chierichetti et al. [Chierichetti et al., 2009b, Chierichetti et al., 2009a] shows that various models (preferential attachment, ACL model, copying, Kronecker multiplication model, Kleinberg's model) are incompressible and suggests a model for the Web graph that complies with the empirical findings of Boldi and Vigna.

Models of real-world networks

Models

1. classic
   - grown versus static random graphs (CHKNS)
   - growth with preferential attachment
   - structure + randomness $\rightarrow$ small-world networks
2. more models
   - Copying model
   - Cooper-Frieze model
   - Kronecker graphs
   - Chung-Lu model
   - Forest-fire model

CHKNS model

Callaway, Hopcroft, Kleinberg, Newman and Strogatz [Callaway et al., 2001]

- simple growth model for a random graph without preferential attachment
- main thesis: grown graphs, however randomly they are constructed, are fundamentally different from their static random-graph counterparts

CHKNS model

- start with 0 vertices at time 0.
- at time t, a new vertex is created
- with probability δ add a random edge by choosing two existing vertices uniformly at random

CHKNS model

let $d_k(t)$ be the number of vertices of degree k at time t

then

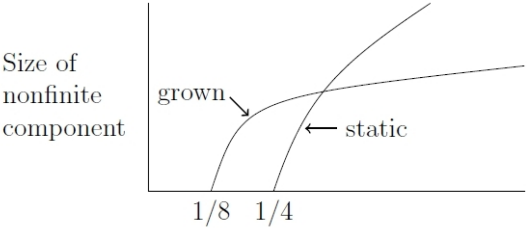
$$\mathbb{E}\left[d_0(t+1)\right] = \mathbb{E}\left[d_0(t)\right] + 1 - \delta \frac{2\mathbb{E}\left[d_0(t)\right]}{t}$$

$$\mathbb{E}\left[d_k(t+1)\right] = \mathbb{E}\left[d_k(t)\right] + \delta\left(\frac{2\mathbb{E}\left[d_{k-1}(t)\right]}{t} - \frac{2\mathbb{E}\left[d_k(t)\right]}{t}\right)$$

it turns out that

$$\frac{\mathbb{E}\left[d_k(t)\right]}{t} = \frac{1}{2\delta+1}\left(\frac{2\delta}{2\delta+1}\right)^k$$

CHKNS model



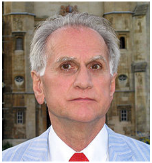
size of giant component for a CHKNS random graph and a static random graph with the same degree distribution

- why are grown and static random graphs so different?
- intuition:
  - positive correlation between the degrees of connected vertices in the grown graph
  - older vertices tend to have higher degree, and to link with other high degree vertices, merely by virtue of their age

Preferential attachment

R. Albert

L. Barabási



B. Bollobás

O. Riordan

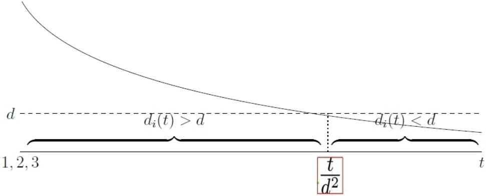
growth model:

- at time n, vertex n is added to the graph
- one edge is attached to the new vertex
- the other vertex is selected at random with probability proportional to its degree
- obtain a sequence of graphs $\{G_1^{(n)}\}$.

Preferential attachment — generalization

- The case of $G_m^{(n)}$ where instead of a single edge we add m edges reduces to $G_1^{(n)}$ by creating a $G_1^{(nm)}$ and then collapsing vertices $km, km-1, \ldots, (k-1)m+1$ to create vertex k.
- An equivalent way of generating $G_m^{(n)}$ is the following: we start with a single vertex consisting of m self-loops. At time t we add a new vertex v_t with m edges adjacent to it. The endpoints of these edges are chosen *sequentially* and preferentially. In other words, after we add each edge, we update the degrees.

Preferential attachment



at time t, vertices 1 to $\frac{1}{d^2}$ have degrees greater than d (Source [Hopcroft and Kannan, 2012])

heuristic analysis

- $\deg_i(t)$ the *expected* degree of the i-th vertex at time t
- the probability an edge is connected to i is $\frac{\deg_i(t)}{2t}$
- therefore

$$\frac{\partial \deg_i(t)}{\partial t} = \frac{\deg_i(t)}{2t}$$

- the solution is $\deg_i(t) = \sqrt{\frac{t}{i}}$

Preferential attachment

$$\int_0^d \mathbf{Pr}\left[\text{degree} = d\right] \partial d = \mathbf{Pr}\left[\text{degree} \le d\right] = 1 - \frac{1}{d^2}$$

by using the fact that $d_i(t) < d$ if $i > \frac{t}{d^2}$ and by taking the derivative

$$\mathbf{Pr}\left[\text{degree} = d\right] = \frac{\partial}{\partial d}\Big(1 - \frac{1}{d^2}\Big) = \frac{2}{d^3}$$

power law distribution!

these results can be proved rigorously using the linearized chord diagrams (LCD) model and also prove strong concentration around the expectation using martingales

Preferential attachment

Theorem

Let $deg_i(t)$ be the degree of vertex i at time t in the preferential attachment model with $m = 1$[a] Then,

$$\mathbb{E}\left[deg_i(t)\right] = \frac{\Gamma(t+1)\Gamma(i-\frac{1}{2})}{\Gamma(t+\frac{1}{2})\Gamma(i)}.$$

where $\Gamma(t) = \int_0^{+\infty} x^{t-1}e^{-x}dx$.

[a] Self-loops contribute 2 to the degree.

Proof.

On whiteboard. □

Preferential attachment

Let $P_k(t) = \frac{1}{t}\sum_{i=1}^{t} 1(deg_i(t) = k)$, $p_k = \frac{2m(m+1)}{k(k+1)(k+2)}$.

Theorem

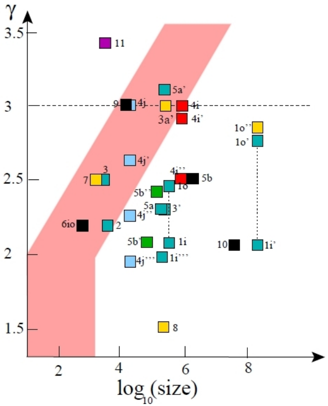
There exists a constant C such that as $t \to +\infty$

$$\mathbf{Pr}\left[\max_k |P_k(t) - p_k(t)| \geq C\sqrt{\frac{\log t}{t}}\right] = o(1).$$

Proof.

On whiteboard. □

Generalized preferential attachment



log-linear plot of the exponents of all the networks reported as having power-law (source [Dorogovtsev and Mendes, 2002])

many real-world networks have a power-law slope $2 < \alpha < 3$

Generalized preferential attachment

how can we tune the power-law slope?

- [Buckley and Osthus, 2004] analyze a modified preferential attachment process where $\alpha > 0$ is a *fitness* parameter
- when t vertex comes in, it chooses i according to

$$\mathbf{Pr}\left[t \text{ chooses } i\right] = \begin{cases} \frac{\deg_{t-1}(i)+\alpha-1}{(\alpha+1)t-1}, & \text{if } 1 \leq i \leq t-1 \\ \frac{\alpha}{(\alpha+1)t-1}, & \text{if } i = t \end{cases}.$$

- $\alpha = 1$ gives the Barabási-Albert/Bollobás-Riordan $G_1^{(n)}$ model
- the power-law slope is $2 + \alpha$.

Generalized preferential attachment

- clustering coefficient of $G_m^{(n)}$ is $\frac{(m-1)\log^2 n}{8n}$ in expectation
- therefore tends to 0 [Bollobás and Riordan, 2003].
- can also be fixed by generalizing the model [Holme and Kim, 2002, Ostroumova et al., 2012].
- triangle formation: if an edge between v and u was added in the previous preferential attachment step, then add one more edge from v to a randomly chosen neighbor of u.

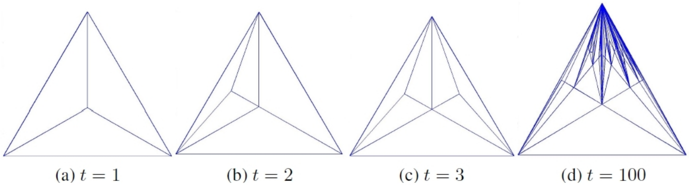
Holme-Kim Model

- perform a preferential attachment step
- the perform with probability β_t another preferential attachment step or a triangle formation step with probability $1-\beta_t$

diameter for PA and GPA is $\frac{\log n}{\log \log n}$ and $\log n$ respectively

Random Apollonian networks

are there power-law planar graphs?

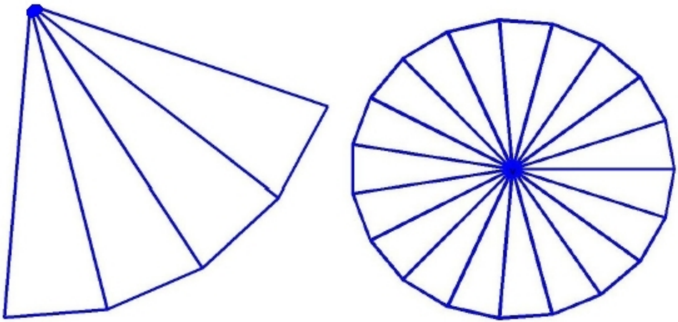


snapshots of a random Apollonian network (RAN) at:
(a) $t = 1$ (b) $t = 2$ (c) $t = 3$ (d) $t = 100$

- at time $t + 1$ we choose a face F uniformly at random among the faces of G_t
- let (i, j, k) be the vertices of F
- we add a new vertex inside F and we connect it to i, j, k

Random Apollonian networks

Preferential attachment mechanism



what each vertex "sees" (boundary and the rest respectively)

Random Apollonian networks

Theorem ([Frieze and Tsourakakis, 2013])

Let $Z_k(t)$ *denote the number of vertices of degree* k *at time* t, $k \geq 3$. *For any* $t \geq 1$ *and any* $k \geq 3$ *there exists a constant* b_k *depending on* k *such that*

$$|\mathbb{E}[Z_k(t)] - b_k t| \leq K, \quad \textit{where} \quad K = 3.6.$$

Furthermore, for t *sufficiently large and any* $\lambda > 0$

$$\mathbf{Pr}\left[|Z_k(t) - \mathbb{E}[Z_k(t)]| \geq \lambda\right] \leq e^{-\frac{\lambda^2}{72t}}$$

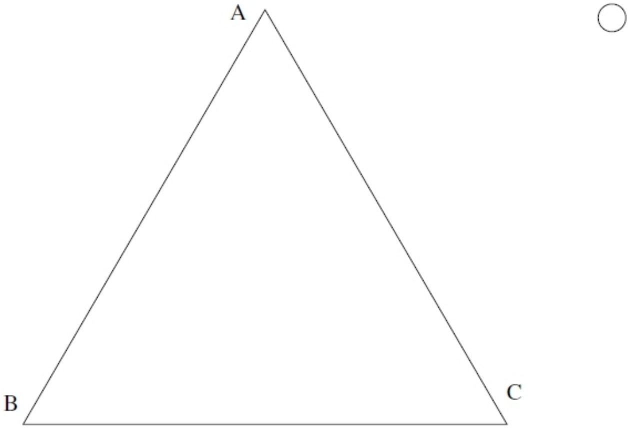
Corollary

The diameter $d(G_t)$ *of* G_t *satisfies asymptotically* **whp**

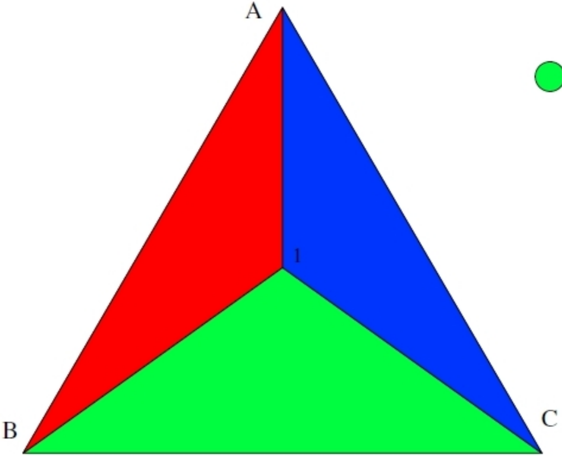
$$\mathbf{Pr}[d(G_t) > 7.1 \log t] \to 0$$

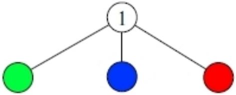
Random Apollonian networks

key idea: establish a bijection with random ternary trees

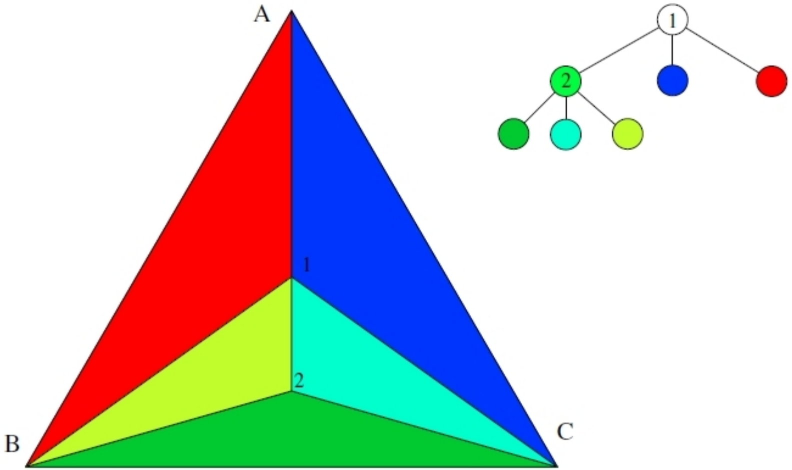


Random Apollonian networks

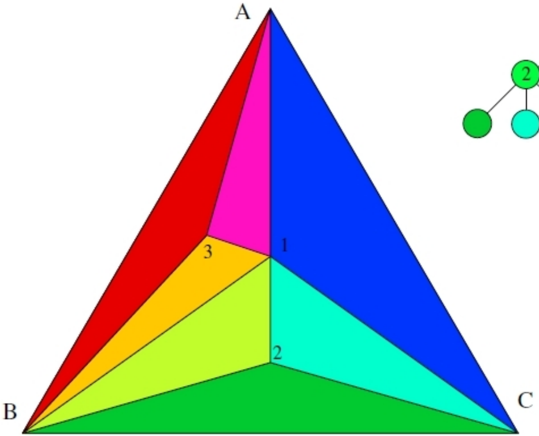


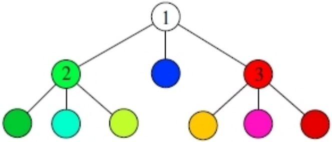


Random Apollonian networks



Random Apollonian networks





Small-world models

Duncan Watts

Steven Strogatz

construct a network with

- small diameter
- positive density of triangles

Small-world models

why should we want to construct a network with

- small diameter,
- positive density of triangles?

$$L(G) = \sum_{\text{pairs} u,v} \frac{d(u,v)}{\binom{n}{2}}, C(G) = \frac{1}{n} \sum_i C_i.$$

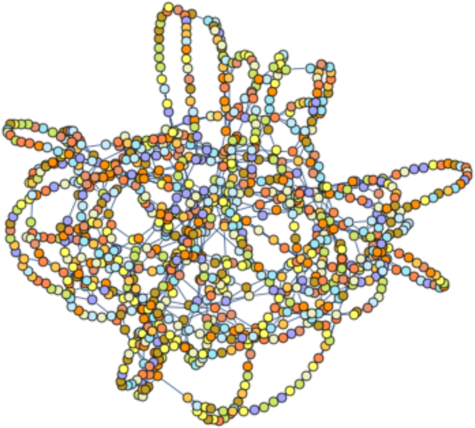
Graph	$\sim |V|$	$2|E|/|V|$	L_{actual}	L_{random}	C_{actual}	C_{random}
Film actors	225K	61	3.65	2.99	0.79	0.00027
Power grid	5K	2.67	18.7	12.4	0.08	0.005
C. elegans	0.3K	14	2.65	2.25	0.28	0.05

Small-world models

model

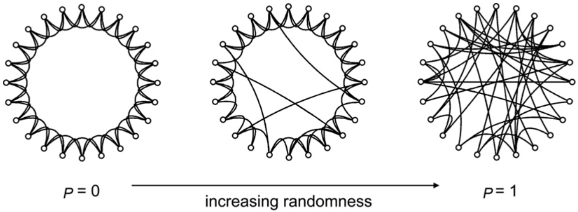
- let G be the r-th power of the cycle on n vertices
- notice that $\mathrm{diam}(G) = \frac{n}{2r}$ and $C(G) = \frac{3(r-1)}{2(2r-1)}$
- let $G(p)$ be the graph obtained from G by deleting independently each edge with probability and then adding the same number of edges back at random

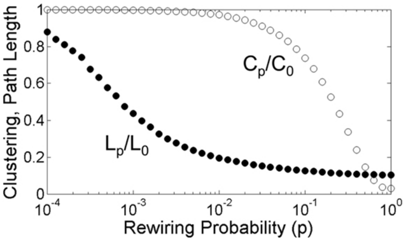
Small-world models



Watts-Strogatz on 1 000 vertices with rewiring probability $p = 0.05$

Small-world models

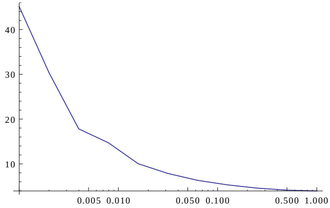




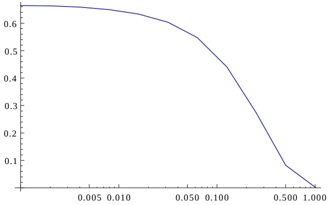
rewiring probability, p

even for a small value of p, $L(G(p))$ drops to $O(\log n)$, which $C(G(p)) \approx \frac{3}{4}$

Small-world models



average distance



clustering coefficient

Watts-Strogatz graph on 4 000 vertices, starting from a 10-regular graph

- intuition: if you add a little bit of randomness to a structured graph, you get the small world effect
- related work: see [Bollobás and Chung, 1988]

Navigation in a small world

Jon Kleinberg

how to find short paths using only local information?

- we will use a simple directed model [Kleinberg, 2000].
- a local algorithm
  - can remember the source, the destination and its current location
  - can query the graph to find the long-distance edge at the current location.

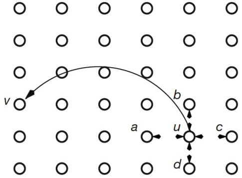
Navigation in a small world

$d(u, v)$: shortest path distance using only original grid edges

directed graph model, parameter r :

- each vertex is connected to its four adjacent vertices
- for each vertex v we add an extra link (v, u) where u is chosen with probability proportional to $d(v, u)^{-r}$

notice: compared to the Watts-Strogatz model the long range edges are added in a biased way



(source [Kleinberg, 2000])

Navigation in a small world

- $r = 0$: random edges, independent of distance
- as r increases the length of the long distance edges decreases in expectation

results

1. $r < 2$: the end points of the long distance edges tend to be uniformly distributed over the vertices of the grid
   - is unlikely on a short path to encounter a long distance edge whose end point is close to the destination
   - no local algorithm can find them
2. $r = 2$: there are short paths
   - a short path can be found be the simple algorithm that always selects the edge that takes closest to the destination
2. $r > 2$: there are no short paths, with high probability

Copying model

[Kumar et al., 2000] analyze the copying model of [Kleinberg et al., 1999b].

- $\alpha \in (0, 1)$: copy factor
- d constant out degree.

evolving copying model, time $t + 1$

- create a new vertex $t + 1$
- choose a prototype vertex $u \in V_t$ uniformly at random
- the i-th out-link of $t + 1$ is chosen as follows:

with probability α we select $x \in V_{t-1}$ uniformly at random, and with the remaining probability it copies the i-th out-lin of u

Copying model

in-degrees follow power-law distribution [Kumar et al., 2000]

Theorem

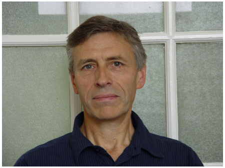
for $r > 0$ *the limit* $P_r = \lim_{t \to +\infty} \frac{N_t(r)}{t}$ *exists and satisfies*

$$P_r = \Theta(r^{-\frac{2-\alpha}{1-\alpha}}).$$

explains the large number of bipartite cliques in the web graph

static models with power-law degree distributions do not account for this phenomenon!

Cooper-Frieze model



Colin Cooper

Alan Frieze

Cooper and Frieze [Cooper and Frieze, 2003] introduce a general model

1. many parameters
2. generalizes preferential attachment, generalized preferential attachment and copying models
3. whose attachment rule is a mixture of preferential and uniform

Cooper-Frieze model

findings

1. we can obtain densification and shrinking diameters
   - add edges among existing vertices
2. power law in expectation and strong concentration under mild assumptions.
3. novel techniques for concentration

   martingales + Laplace

Kronecker graphs

reminder: Kronecker product

$A = [a_{ij}]$ an $m \times n$ matrix

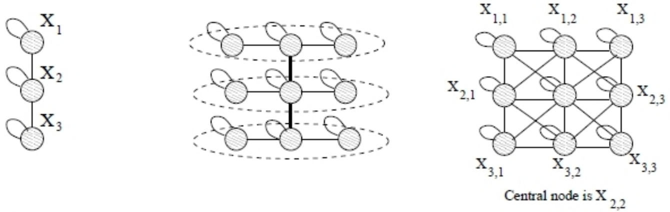
$B = [b_{ij}]$ a $p \times q$ matrix

then, $A \otimes B$ is the $mp \times nq$ matrix

$$\begin{pmatrix} a_{11}B & .. & a_{1n}B \\ .. & .. & .. \\ a_{m1}B & .. & a_{mn}B \end{pmatrix}$$

[Leskovec et al., 2010] propose a model based on the Kronecker product, generalizing RMAT [Chakrabarti et al., 2004].

Kronecker graphs



(a) Graph K_1 (b) Intermediate stage (c) Graph $K_2 = K_1 \otimes K_1$

1	1	0
1	1	1
0	1	1

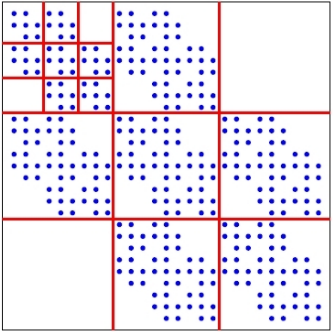
(d) Adjacency matrix of K_1

K_1	K_1	0
K_1	K_1	K_1
0	K_1	K_1

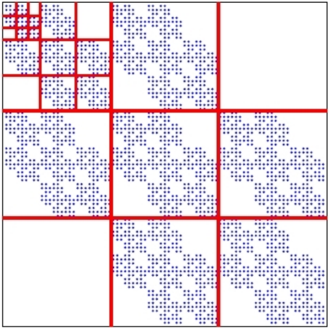
(e) Adjacency matrix of $K_2 = K_1 \otimes K_1$

source [Leskovec et al., 2010]

Kronecker graphs



(a) K_3 adjacency matrix (27×27)



(b) K_4 adjacency matrix (81×81)

source [Leskovec et al., 2010]

Kronecker graphs

a stochastic Kronecker graph is defined by two parameters

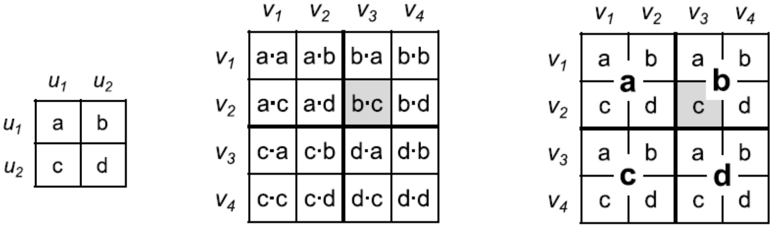
- an integer k
- the seed/initiator matrix θ

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

- we obtain a graph with $n = 2^k$ vertices by taking repeatedly Kronecker products
- let $A_{k,\theta} = \underbrace{\theta \otimes \ldots \otimes \theta}_{l \text{ times}}$ be the resulting matrix
- adjacency matrix $\bar{A}_{k,\theta}$ obtained by a randomized rounding
- typically 2×2 seed matrices are used;

  however, one can use other seed matrices

Kronecker graphs



in practice we never need to compute A, but we can actually do a sampling based on the hierarchical properties of Kronecker products.

Kronecker graphs

consider $G(V, E)$ such that $|V| = n = 2^k$.

- Erdős-Rényi

$$\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

- core-periphery

$$\begin{pmatrix} 0.9 & 0.5 \\ 0.5 & 0.1 \end{pmatrix}$$

- hierarchical community structure

$$\begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$$

Kronecker graphs

- power-law degree distributions [Leskovec et al., 2010]
- power-law eigenvalue distribution [Leskovec et al., 2010]
- small diameter [Leskovec et al., 2010]
- densification power law [Leskovec et al., 2010]
- shrinking diameter [Leskovec et al., 2010]
- triangles [Tsourakakis, 2008]
- connectivity [Mahdian and Xu, 2007]
- giant components [Mahdian and Xu, 2007]
- diameter [Mahdian and Xu, 2007]
- searchability [Mahdian and Xu, 2007]

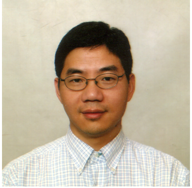
Kronecker graphs

how do we find a seed matrix θ such that $A_G \approx \underbrace{\theta \otimes \ldots \otimes \theta}_{k \text{ times}}$?

- maximum-likelihood estimation: $argmax_{\theta} \mathbf{Pr}[G|\theta]$
- hard since exact computation requires $O(n!n^2)$ time, but
- Metropolis sampling and approximations allow $O(m)$ time good approximations [Leskovec and Faloutsos, 2007]
- moment based estimation: express the expected number of certain subgraphs (e.g., edges, triangles, triples) as a function of a, b, c and solve a system of equations [Gleich and Owen, 2012]

Chung-Lu model

Fan Chung Graham



Linyuan Lu

- model is specified by $w = (w_1, \ldots, w_n)$ representing expected degree sequence
- certices i, j are connected with probability

$$p_{ij} = \frac{w_i w_j}{\sum_{k=1}^{n} w_k} = \rho w_i w_j.$$

- to have a proper probability distribution $w_{\max}^2 \leq \rho$
- can obtain an Erdős-Rényi random graph by setting

$$w = (pn, \ldots, pn)$$

Chung-Lu model

how to set the weights to get power law exponent β?

- the probability of having degree k in power law

$$\mathbf{Pr}\left[\deg(v) = k\right] = \frac{k^{-\beta}}{\zeta(\beta)}$$

- hence, for $\beta > 1$

$$\mathbf{Pr}\left[\deg(v) \geq k\right] = \sum_{l \geq k}^{+\infty} \frac{k^{-\beta}}{\zeta(\beta)} = \frac{1}{\zeta(\beta)(\beta-1)k^{\beta-1}}$$

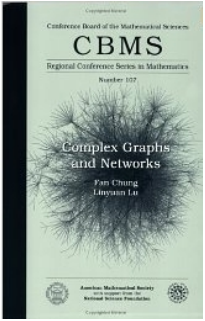
- assuming weights are decreasing and setting $w_i = k,\ i/n = \mathbf{Pr}\left[\deg(v) \geq k\right]$

$$w_i = \left(\frac{i}{\zeta(\beta)(\beta-1)i}\right)^{-\frac{1}{\beta-1}}$$

Chung-Lu model

rigorous results on:

- degree sequence
- giant component
- average distance and the diameter
- eigenvalues of the adjacency and the Laplacian matrix
- ...

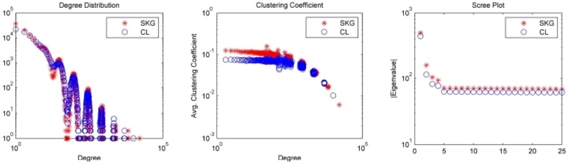


Complex graphs and networks, AMS

Kronecker vs. Chung-Lu

“the SKG model is close enough to its associated CL model that most users of SKG could just as well use the CL model for generating graphs.”

[Pinar et al., 2011]



(a) Degree distribution (b) Clustering coefficients (c) Eigenvalues of adjacency matrices

Comparison of the graph properties of SKG and an equivalent CL.

Forest-fire model

J. Leskovec

J. Kleinberg

C. Faloutsos

[Leskovec et al., 2007] propose the forest fire model that is able to re-produce at a qualitative scale most of the established properties of real-world networks

Forest-fire model

basic version of the model

1. p : forward burning probability
2. r : backward burning ratio

- initially, we have a single vertex
- at time t a new vertex v arrives to G_t
  - node v picks an *ambassador/seed* node u uniformly at random link to u
  - two numbers x, y are sampled from two geometric distributions with parameters $\frac{p}{1-p}$ and $\frac{rp}{1-rp}$ respectively
  - then, v chooses x out-links and y in-links of u which are incident to unvisited vertices
  - let $u_1, \ldots, u_{x+y}$ be these chosen endpoints
  - mark $u_1, \ldots, u_{x+y}$ as visited and apply the previous step recursively to each of them

Forest-fire model

(Few) Remarks

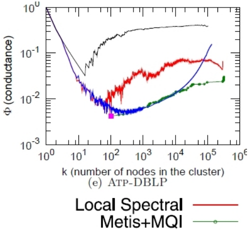
- There is a "flavor" of both preferential attachment and a copying mechanism.
- The number of edges of an incoming vertex can vary a lot, depending on its *ambassador*.
- We can have small fires but also large fires

Forest-fire model

the forest-fire model is able to explain

- heavy tailed in-degrees and out-degrees
- densification power law
- shrinking diameter
- ...
- deep cuts at small size scales and the absence of deep cuts at large size scales

reminder



NCP of a DBLP graph (source [Leskovec et al., 2009]).

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