

Lecture 23

Log-concavity for $t_M(x,y,z,w) = (x+y)^{-1} (y+z)^r (x+w)^{|E|+r} T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right)$.

$$\left(= \sum_{ijkl} \left(\int_{X_E} \alpha^i \beta^j c_k(S_M^\vee) c_l(Q_M) x^i y^j z^k w^l \right) \right)$$

Recall: $0 \rightarrow S_L \rightarrow \underline{C}^E \rightarrow Q_L \rightarrow 0$

Also, $[S_M] + [Q_M] = [C^E] = |E| \cdot 1 \in K(X_E)$

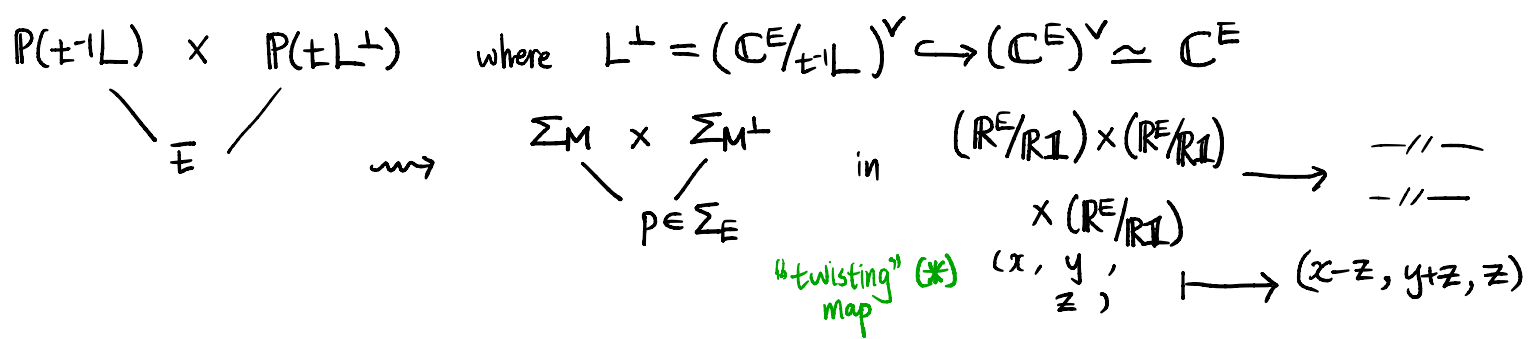
Thus, $s(Q_M^\vee) = c(S_M^\vee)$ and $s(S_M) = c(Q_M)$.

Consider $\tilde{X} = \mathbb{P}_{X_E}(S_L) \times \mathbb{P}_{X_E}(Q_L^\vee) = \text{Biproj}_{X_E}(Sym^* S_L^\vee \otimes Sym^* Q_L) \hookrightarrow X_E \times \mathbb{P}^n \times \mathbb{P}^n$
 and let $c_1(\mathcal{O}_{(0,0)}) = \delta$, $c_1(\mathcal{O}_{(0,1)}) = \eta$.

Have $\pi_* (\delta^{r-1+l} \eta^{n-r+k}) = s_r(S_L) s_k(Q_L^\vee) = c_k(S_L^\vee) c_l(Q_L)$.

$\Rightarrow \int_{\tilde{X}} \pi^* \alpha^i \cdot \pi^* \beta^j \cdot \delta^{r-1+l} \eta^{n-r+k} = \int_X \alpha^i \beta^j c_k(S_L) c_l(Q_L)$

\Rightarrow Prop $t_M(x,y,z,w)$ is a denormalized Lorentzian polynomial.



Thm $\exists \Sigma \subset (\mathbb{R}^E/\mathbb{R}1)^3$ with $\tilde{\phi}: X_\Sigma \xrightarrow{\text{bivat.}} X_E \times \mathbb{P}^n \times \mathbb{P}^n$ with underlying map of Cochar as in (*) such that for any M , Σ has a subfan whose supp. is $\Sigma_E \times \Sigma_M \times \Sigma_{M^\perp}$, with the property that

$$\tilde{\phi}_* ([\Sigma_E \times \Sigma_M \times \Sigma_{M^\perp}] \cdot \delta^{r-1+l} \eta^{n-r+k}) = c_k(S_M^\vee) c_l(Q_M)$$

where $\delta = \tilde{\phi}^*[H_1]$, $\eta = \tilde{\phi}^*[H_2]$ for $H_1 \subset \text{first } \mathbb{P}^n$, $H_2 \subset \text{second } \mathbb{P}^n$.

Rem Trop. model of $P_{W_L}(Q_L)$ is $\Sigma_M \times \Sigma_M^+$.
——//—— $P_{W_L}(S_L)$ is $\Sigma_M \times \Sigma_M$.

Proof of thm crucially uses the valuativity property of S_M & Q_M .

Cor $t_M(x, y, z, w)$ is a denormalized Lorentzian polynomial.

Rem Similar game for flag matroids.

N.B. $\chi(W_L; \mathcal{O}_{X_E}(D_{\text{PCM}})|_{W_L}) = h^0(W_L; \mathcal{O}(D_{\text{PCM}})|_{W_L}) = T_M(1, 0)$

Q. Is $H^i(X_E; \Lambda^j Q_L) = 0 \quad \forall j \geq 0 \text{ and } i > 0$?

(Matroid Borel-Weil-Bott?)

Depends on the realization L ? char?