Lecture 23

Log-concavity for $t_M(x,y,z,w) = (x+y)^{-1}(y+z)^r(x+w)^{|E|-r}\, T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right).$

$$\left(= \sum_{i,j,k,l} \left(\int_{X_E} \alpha^i \beta^j c_k(S_M^\vee)\, c_l(Q_M)\right) x^i y^j z^k w^l\right)$$

Recall: $0 \to S_L \to \underline{\mathbb{C}}^E \to Q_L \to 0$

Also, $[S_M] + [Q_M] = [\underline{\mathbb{C}}^E] = |E|\cdot 1 \in K(X_E)$

Thus, $s(Q_M^\vee) = c(S_M^\vee)$ and $s(S_M) = c(Q_M)$.

Consider $\widetilde{X} = \mathbb{P}_{X_E}(S_L) \times \mathbb{P}_{X_E}(Q_L^\vee) = \mathrm{Biproj}_{X_E}(\mathrm{Sym}^\bullet S_L^\vee \otimes \mathrm{Sym}^\bullet Q_L) \hookrightarrow X_E \times \mathbb{P}^n \times \mathbb{P}^n$

and let $c_1(\mathcal{O}(1,0)) = \delta$, $c_1(\mathcal{O}(0,1)) = \eta$.

Have $\pi_*(\delta^{r-1+l}\eta^{n-r-k}) = s_l(S_L)\, s_k(Q_L^\vee) = c_k(S_L^\vee)\, c_l(Q_L)$.

$$\Rightarrow \int_{\widetilde{X}} \pi^*\alpha^i \cdot \pi^*\beta^j \cdot \delta^{r-1+l}\eta^{n-r+k} = \int_X \alpha^i\beta^j c_k(S_L)\, c_l(Q_L)$$

$\Rightarrow$ <u>Prop</u> $t_{M(L)}(x,y,z,w)$ is a denormalized Lorentzian polynom.

$$\begin{tikzcd}[column sep=small] \mathbb{P}(t^{-1}L) \arrow[dr, dash] & \times & \mathbb{P}(tL^\perp) \arrow[dl, dash] \\ & E & \end{tikzcd}$$

where $L^\perp = (\mathbb{C}^E/t^{-1}L)^\vee \hookrightarrow (\mathbb{C}^E)^\vee \simeq \mathbb{C}^E$

$$\rightsquigarrow \quad \begin{tikzcd}[column sep=small] \Sigma_M \arrow[dr, dash] & \times & \Sigma_{M^\perp} \arrow[dl, dash] \\ & p \in \Sigma_E & \end{tikzcd} \quad \text{in} \quad (\mathbb{R}^E/\mathbb{R}\mathbb{1}) \times (\mathbb{R}^E/\mathbb{R}\mathbb{1}) \times (\mathbb{R}^E/\mathbb{R}\mathbb{1}) \longrightarrow \text{—//—}\ \text{—//—}$$

"twisting" map (✱) $\quad (x, y, z) \longmapsto (x-z, y+z, z)$

<u>Thm</u> $\exists \Sigma \subset (\mathbb{R}^E/\mathbb{R}\mathbb{1})^3$ with $\widetilde{\phi}: X_\Sigma \xrightarrow{\text{birat.}} X_E \times \mathbb{P}^n \times \mathbb{P}^n$ with underlying map of Cochar as in (✱) such that for any M, Σ has a subfan whose supp. is $\Sigma_E \times \Sigma_M \times \Sigma_{M^\perp}$, with the property that

$$\widetilde{\phi}_*\left([\Sigma_E \times \Sigma_M \times \Sigma_{M^\perp}] \cdot \delta^{r-1+i}\eta^{n-r+k}\right) = c_k(S_M^\vee)\, c_l(Q_M)$$

where $\delta = \widetilde{\phi}^*[H_1]$, $\eta = \widetilde{\phi}^*[H_2]$ for $H_1 \subset$ first $\mathbb{P}^n$, $H_2 \subset$ second $\mathbb{P}^n$.

Rem Trop. model of $\mathbb{P}_{W_L}(Q_L^\vee)$ is $\Sigma_M \times \Sigma_{M^\perp}$.

—— // —— $\mathbb{P}_{W_L}(S_L)$ is $\Sigma_M \times \Sigma_M$.

Proof of thm crucially uses the valuativity property of S_M & Q_M.

Cor $t_M(x,y,z,w)$ is a denormalized Lorentzian polynomial.

Rem Similar game for flag matroids.

N.B. $\chi(W_L ; \mathcal{O}_{X_E}(D_{-P(M)})|_{W_L}) = h^0(W_L ; \mathcal{O}(D_{-P(M)})|_{W_L}) = T_M(1,0)$

Q. Is $H^i(X_E ; \Lambda^j Q_L) = 0$ $\forall j \geq 0$ and $i > 0$?

(Matroid Borel-Weil-Bott?)

Depends on the realization L? char?