Lecture 15 (As usual, M, N dual lattices of rank n)

Recall: $\Sigma \subset N_{\mathbb{R}}$ a smth proj. fan (w/ $\mathrm{lin}(\Sigma) = 0$) $\leftrightsquigarrow$ X_Σ a T_N-toric variety

Σ
$\curlyvee\!\mid$
$\Sigma' \rightsquigarrow X_\Sigma \longrightarrow X_{\Sigma'}$ "blow-up"

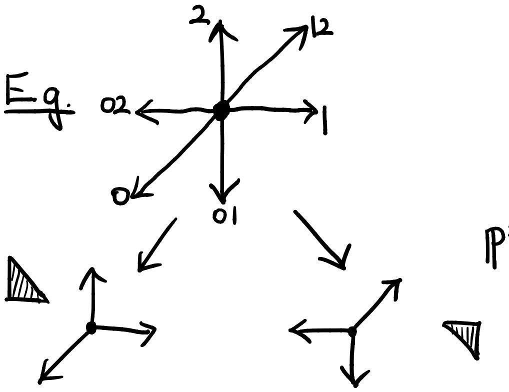
$\sigma \in \Sigma(k) \leftrightsquigarrow V(\sigma) \subseteq X_\Sigma$ codim k orbit closure

$= \bigcap_{\rho \preceq \sigma} D_\rho \quad (D_\rho = V(\rho),\ \rho \in \Sigma(1))$

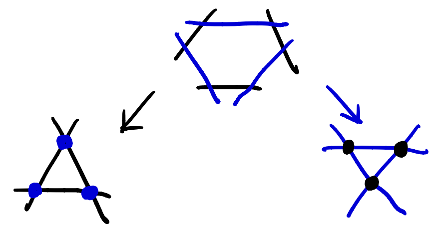
$\{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -\varphi_\rho \forall \rho\} = P_D \in \mathrm{Def}(\Sigma) \leftrightsquigarrow \mathcal{O}_{X_\Sigma}(D = \sum_\rho \varphi_\rho D_\rho)$ b.p.f.

$X_\Sigma \longrightarrow \mathbb{P}(\mathbb{C}^{P_D \cap M})$

E.g.



$$\begin{tikzcd} & X_{A_2} = Bl_{3pts}\, \mathbb{P}^2 \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\text{crem}"] & & \mathbb{P}^2 \end{tikzcd}$$



Defn The <u>K-ring</u> $K(X)$ of a smth variety X is the free ab. grp. gen. on vector bnd's on X modulo short exact sequences.

$$\mathbb{Z}\{[\mathcal{E}] \mid \mathcal{E} \in \mathrm{Vect}(X)\} \Big/ \langle [\mathcal{E}] - [\mathcal{E}'] - [\mathcal{E}''] \mid 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \rangle$$

E.g. $K(pt) = \mathbb{Z}$.

$\chi : K(X) \longrightarrow \mathbb{Z}$ the sheaf Euler char.

Defn For a (smooth) lattice polytope P, define a ring $\mathbb{I}(P)$ by:

$$\mathbb{Z}\{[Q] \mid Q \in \mathrm{Def}(P)\} \big/ \langle [Q] = [Q+m] \rangle + \langle \ker \text{ of } [Q] \mapsto 1_Q \rangle$$

(often called <u>McMullen's polytope algebra</u>)

Thm $\mathbb{I}(P) \xrightarrow{\simeq} K(X_{\Sigma_P})$ via $[Q] \mapsto [\mathcal{O}_{X_{\Sigma_P}}(D_Q)]$.

$$\begin{tikzcd} \mathbb{I}(P) \arrow[rr, "\simeq"] \arrow[dr] & & K(X_{\Sigma_P}) \arrow[dl, "\chi"] \\ & \mathbb{Z} & \end{tikzcd}$$

$\chi(\mathcal{O}(D_Q)) = \#(Q \cap M)$

pf) [Morelli] $+ \alpha$

Thm [Cameron-Fink][Bernardi-Kalman-Postnikov] Let $\Psi : \mathbb{Q}[t,u] \to \mathbb{Q}[x,y]$ $\binom{t}{i}\binom{u}{j} \mapsto x^i y^j$.

$$\Psi\left(\chi\left([P(M)] + t\nabla + u\nabla\right)\right) = (x+y-1)^{-1} y^r x^{|E|-r} T_M\left(\frac{x+y-1}{y}, \frac{x+y-1}{x}\right)$$

Thm (Fink-Speyer '12) Consider

$$\begin{array}{ccc} & Fl(1,r,n;E) & \\ {}^{\pi_{1r}}\swarrow & & \searrow^{\pi_{1n}} \\ Gr(r;E) & & \mathbb{P}^n \times \mathbb{P}^n \end{array}$$

For $L \in Gr(r;E)$ realizing M, have $(\pi_{1n})_* \pi_{1r}^*(\mathcal{O}_{\overline{T\cdot L}}(1)) = T_M(x,y)$

$$\in \frac{\mathbb{Z}[x,y]}{\langle x^{n+1}, y^{n+1}\rangle} \simeq K(\mathbb{P}^n \times \mathbb{P}^n)$$

Rem (Dinu-E.-Seynnaeve '21) This for flag matroids.

Thm There are isomorphisms (via $\eta_{V(\sigma)} \mapsto x_\sigma := \prod_{\rho \leq \sigma} x_\rho$)

$$H^{2\bullet}(X_\Sigma) \simeq A^\bullet(X_\Sigma) \simeq \frac{\mathbb{Z}[x_\rho \mid \rho \in \Sigma(1)]}{\left\langle \prod_{\rho \in I} x_\rho \mid I \subseteq \Sigma(1) \text{ does not form a cone} \right\rangle + \left\langle \sum_{\rho \in \Sigma(1)} \langle u_\rho, m\rangle x_\rho \mid m \in M \right\rangle}$$

Rem Can consider above as piecewise polynom. ring modulo the ideal gen. by global polynom

Rem Definition of $A^\bullet(\Sigma)$ as a quotient of $\mathbb{Z}[x_\rho\text{'s}]$ make sense for any (possibly incomplete) fan.

$$\begin{cases} A^\bullet(X_\Sigma) \text{ satisfies Poincare duality with } \int_{X_\Sigma} : A^n(X_\Sigma) \to \mathbb{Z}, \quad x_\sigma \mapsto 1 \quad \forall \sigma \in \Sigma(n) \\ A^k(X_\Sigma) \text{ gen. by } \{x_\sigma \mid \sigma \in \Sigma(k)\}. \end{cases}$$

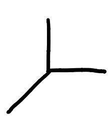
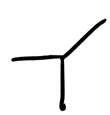
$\Downarrow$

Defn / Prop A <u>Minkowski weight</u> of dim. k, denoted $\Delta \in MW_k(\Sigma)$, is a fct $\Delta : \Sigma(k) \to \mathbb{Z}$ st for any $\tau \in \Sigma(k-1)$,

$$\sum_{\tau < \sigma} \Delta(\sigma) u_{\sigma \setminus \tau} \in \operatorname{span}(\tau).$$

Thm $MW_\bullet(\Sigma)$ has the ring str. given by <u>stable intersection</u> $\Delta \cap_{st} \Delta'$.

[Fulton-Sturmfels '97]

Eg.  $\cap_{st}$  $=$ $\cdot 2$