A brief note on McMullen's g-conjecture

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Abstract

We give a brief account of the two proofs of McMullen's g-conjecture characterizing the f-vectors of simple (simplicial) polytopes. The sufficiency of the conditions outlined in the g-conjecture was proved by Billera and Lee. The necessity part was first proved by Stanley using Hodge theory on intersection cohomology of simplicial toric varieties. Later McMullen developed Hodge theory on the polytope algebra which gives a proof of the necessity without appealing to algebraic geometry.

1 Preliminaries

A **polytope** is a convex full of finitely many points in $\mathbb{R}^d$. We will assume that the polytopes are always full-dimensional in their ambient spaces. The **(inner) normal fan** Σ_P of a polytope $P \subset \mathbb{R}^d$ has cones whose interiors correspond to vectors $\vec{v} \in \mathbb{R}^d$ with the same $\min_{x \in P}\langle v, x\rangle$. A polytope is **simple** if its normal fan is simplicial (i.e. rays of any maximal cone are linearly independent). Dually, a polytope is **simplicial** if all of its faces are simplicies. These are dual notions via the polytope dual: Translating $P \subset \mathbb{R}^d$ if necessary, let $P \ni 0$; then its dual is defined as $P^\circ := \{y \in \mathbb{R}^d \mid \langle y, P\rangle \geq -1\}$.

The f**-vector** $(f_0(P), \ldots, f_d(P))$ of a (simple) polytope $P \subset \mathbb{R}^d$ is

$$f_i := \#(\text{number of faces of dim} = i),$$

where $f_{-1} := 1$. The f-vector of a fan Σ is likewise $f_i(\Sigma) := \#(\text{number of cones of } \textit{codimension } i)$, so that the f-vector of P is the same as the f-vector of its normal fan.

Example 1.1. The 3-cube P is polytope-dual to the octahedron Q. Their f-vectors (including f_{-1}) are $(1, 8, 12, 6, 1)$ and $(1, 6, 12, 8, 1)$. Their f-vectors are mirrors of each other.

Remark 1.2. As our main problem concerns only the f-vectors of polytopes, we'll more or less only work with simple polytopes, as their f-vectors are in bijection with those of simplicial polytopes via duality.

The h**-vector** $(h_0(P), \ldots, h_d(P))$ is obtained from the f-vector by setting $f(t) := \sum_{i=0}^{\infty} f_i t^i$ and

$$h(t) := \sum_{i=0}^{\infty} h_i t^i = f(t-1),$$

or equivalently $h_i := \sum_{j=i}^{d} (-1)^{j-i}\binom{j}{i} f_j$. Often we set $h_{-1} := 0$. An important property of the h-vector is the following.

Theorem 1.3 (Dehn-Sommerville)**.** The h-vector of a simple polytope is palindromic. That is, if $P \subset \mathbb{R}^d$ is a simple polytope, then $h_i(P) = h_{d-i}(P)$ for all $i = 0, \dots, d$.

Example 1.4. If P is the 3-cube again, then its h-polynomial is $h(t) = (t-1)^3 + 6(t-1)^2 + 12(t-1) + 8 = t^3 + 3t^2 + 3t + 1 = (t+1)^3$. The h-vector is palindromic (1,3,3,1). Note that Dehn-Sommerville relation need not hold for non-simple polytopes. For example, the h-polynomial of the octahedron is $h(t) = (t-1)^3 + 8(t-1)^2 + 12(t-1) + 6 = t^3 + 5t^2 - t + 1$.

The g**-vector** is obtained from the h-vector by

$$g(t) = (1-t)h(t) = (1-t)f(t-1),$$

or equivalently $g_i := h_i - h_{i-1}$. A sequence $(a_1, \dots, a_m)$ of non-negative integers is a **Macaulay vector** if the following is satisfied: if $a_i = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_{r_i}}{r_i}$ then $a_{i+1} \leq \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_{r_i}+1}{r_i+1}$. We can now state McMullen's g-conjecture:

Theorem 1.5 (McMullen's g-conjecture)**.** A sequence of non-negative integers $(f_0, \dots, f_d)$ is a f-vector of a (d-dimensional) simple polytope if and only if:

1. $h_i = h_{d-i}$ for all $i = 0, \dots, \lfloor d/2 \rfloor$ (Dehn-Sommerville),
2. $g_i \geq 0$ for all $i = 0, \dots, \lfloor d/2 \rfloor$ (lower bound conjecture), and
3. $(g_1, \dots, g_{\lfloor d/2 \rfloor})$ is a Macaulay vector.

The sufficiency part of the g-conjecture was solved by Billera and Lee ([BL81]). A vague sketch is: f-vector satisfying the above gives a lex-order-ideal, which defines a simplicial complex with the same f-vector, which then can be used to consider some shellable union of facets of a cyclic polytope whose boundary structure reflects the simplicial complex, and then by selecting a general enough point to make a cone then take hyperplane intersection to arrive at the polytope.

In the next two sections we discuss solving the necessity part. While all known proofs rely on some sort of Hodge structure, there are two different approaches to developing the Hodge theory: first the geometric approach by Stanley using toric geometry, and second a purely combinatorial one given by McMullen via polytope algebras.

2 Through toric geometry

Here we follow exposition given in [Ful93] for Stanley's proof of the necessity of the conditions in the g-conjecture. All toric varieties are assumed to be over the complex numbers $\mathbb{C}$.

Two harmless assumptions

First, we may assume that the polytopes are lattice polytopes (i.e. the vertices in $\mathbb{Z}^d$). Two polytopes are **combinatorially equivalent** if they have the same face poset structure. A rational realization of a polytope $P \subset \mathbb{R}^d$ is a polytope P' that is combinatorially equivalent to P and has vertices in $\mathbb{Q}^d$. *Caution: Not* every polytope has a rational realization for a nice family of examples and related rationality issues see [Zie08]. However, simple or simplicial polytopes always do. For simplicial polytope, just wiggle the vertices a bit to make them rational; as the faces are all simplices the face poset does not change. The same goes for simple polytopes, except now we (dually) wiggle the facet hyperplane a bit to be rational. Thus we *assume that our simple polytopes are lattice polytopes.*

Second, we may further assume that Σ_P is smooth. Let $P \subset \mathbb{R}^d$ be a simple lattice polytope. Then the toric variety X_{Σ_P} is an orbifold (locally a quotient of a smooth manifold by a finite group). For orbifolds, the (co)homologies with rational coefficients coincide with the intersection (co)homologies, on which the Kähler package holds for projective varieties. Hence, throughout this section we assume that Σ_P is smooth, with the understanding that for simplicial cases one uses the same arguments by replacing "cohomology" with "intersection cohomology."

h-vector is the Betti numbers

For any complex algebraic variety X of dimension d, one can define the **virtual Poincaré polynomial** $P_X(t)$ characterized by

1. $P_X(t) = \sum_{i=0}^{2d} h^i(X)t^i$ for X complete and smooth, where $h^i(X) := \dim_{\mathbb{Q}} H^i(X;\mathbb{Z}) \otimes \mathbb{Q}$,
2. If $Y \subset X$ is a closed subvariety, then $P_X(t) = P_Y(t) + P_{X\setminus Y}(t)$.

In other words, $P_{(\cdot)}(t)$ is a map from the Grothendieck ring of varieties $K(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[t]$.

From (2), one can deduce from the torus orbit stratification of X_Σ that $P_{X_\Sigma}(t) = f_c(\Sigma)P_{(\mathbb{C}^*)^c}(t)$, and from (1) and Kunnuth formula that $P_{(\mathbb{C}^*)^c}(t) = (t^2-1)^c$ (note $\mathbb{C}^* = \mathbb{P}^1_{\mathbb{C}} \setminus \{0,\infty\}$ so that $P_{\mathbb{C}^*}(t) = t^2 - 1$). Putting these together, we have that for Σ_P smooth, we have

$$P_{X_\Sigma}(t) = \sum_{i=0}^{d} f_i(P)(t^2-1)^i = f_P(t^2-1).$$

In other words, we have $h_i(P) = h^{2i}(X)$. By Poincaré duality on complex manifolds, we thus immediately obtain the Dehn-Sommerville relation $h_i(P) = h_{d-i}(P)$. Moreover, hard Lefschetz property implies that $h^j(X) - h^i(X) \geq 0$ for $0 \leq i \leq j \leq d$, and hence $h_i(P) - h_{i-1}(P) \geq 0$ for $i \leq \lfloor d/2 \rfloor$. Hence, we have proven the necessity of conditions (1) and (2).

Lastly, let $A = A^\bullet(X_P)_{\mathbb{Q}} = H^{2\bullet}(X;\mathbb{Q})$ be the Chow ring, and $\ell \in A^1$ an ample class. Then $R^\bullet := A/\ell$ is a graded $\mathbb{Q}$-algebra generated in degree 1, where $\dim R^i = \dim A^i - \dim A^{i-1} = h_i - h_{i-1} = g_i$ (again by hard Lefschetz). Macaulay described all the dimensions of R^i for such rings, which is what is described in condition (3).

3 McMullen's polytope algebra

Here we follow [McM93] and [Huh16] for McMullen's proof of the g-conjecture via the polytope algebra that does not appeal to hard Lefschetz theorem of intersection cohomologies. The broad approach is similar in that McMullen develops an appropriate Hodge theory on a graded $\mathbb{R}$-algebra whose Hilbert function is the h-vector.

The polytope algebra

Let $V = \mathbb{R}^d$ be fixed throughout this section, and let $\mathscr{P} = \mathscr{P}(V)$ be the set of all polytopes in V. A map $\phi : \mathscr{P} \to A$ where A is an abelian group is **valuative** if for any $P, Q \in \mathscr{P}$ such that $P \cup Q \in \mathscr{P}$ one has

$$\phi(P) + \phi(Q) = \phi(P \cup Q) + \phi(P \cap Q).$$

Moreover, a valuation ϕ is translation invariant if $\phi(Q) = \phi(Q + t)$ for any $t \in V$. The polytope algebra is defined as a universal object with respect to translation invariant valuations as follows.

Definition 3.1. Given a map $\phi : \mathscr{P} \to A$, denote also by $\phi : \mathbb{Z}^{\oplus \mathscr{P}} \to A$ the map ϕ induces. The **polytope algebra** Π is a quotient $\pi : \mathbb{Z}^{\oplus \mathscr{P}} \twoheadrightarrow \Pi$ defined by the universal property that for any translation invariant valuative map $\phi : \mathscr{P} \to A$ there exists a unique map $\widetilde{\phi} : \Pi \to A$ making the following diagram commute.

$$\begin{array}{ccc} \mathbb{Z}^{\oplus \mathscr{P}} & \xrightarrow{\ \phi\ } & A \\ {\scriptstyle \pi}\big\downarrow & \nearrow_{\exists ! \widetilde{\phi}} & \\ \Pi & & \end{array}$$

The condition that $P \cup Q$ be also convex can be slightly cumbersome at time, but letting $U(\mathscr{P}) = \{\text{finite unions of elements of } \mathscr{P}\}$, we have a useful fact ([McM89, Lemma 4]) that valuative maps on $\mathscr{P}$ extend uniquely to $U(\mathscr{P})$. The proof relies on another useful fact ([McM89, Lemma 3]) that $\phi : \mathscr{P} \to A$ is a valuation iff it induces a homomorphism on $X(\mathscr{P})$ (algebra generated by indicator functions χ_P). One consequence of the these is that one can work with $P_1 \cup \cdots \cup P_n$ via inclusion-exclusion principle, and in particular, as any polytope has a triangulation, we have that Π is generated by classes of simplices.

Π has a multiplication structure via Minkowski sum, i.e. $[P_1] \cdot [P_2] := [P_1 + P_2]$, which makes Π into a commutative ring with $[\emptyset] = 0$ and $[pt] = 1$. Moreover, for $\lambda \geq 0$ we have a ring homomorphism $\Delta(\lambda) : [P] \mapsto [\lambda P]$ (dilation). *Caution:* $\lambda[P] \neq [\lambda P]$. The first main theorem on the structure of Π is the follwoing.

Theorem 3.2. [McM89, Theorem 1] Π is a almost graded $\mathbb{R}$-algebra in the following sense. There is a decomposition (as $\mathbb{Z}$-modules) $\Pi = \bigoplus_{r=0}^{d} \Xi_r$ such that

1. $\Xi_0 \simeq \mathbb{Z}$ and Ξ_r for $r > 0$ has a structure of $\mathbb{R}$-vector space, such that
2. $\Xi_i \cdot \Xi_j \subset \Xi_{i+j}$ for $0 \leq i, j \leq d$ ($\Xi_{r>d} = 0$) and $(\lambda x)y = \lambda(xy) = x(\lambda y)$ for $x, y \in \bigoplus_{r>0} \Xi_r$, and
3. $\Delta(\lambda)x = \lambda^r x$ for $x \in \Xi_r$.

First, one notes that there is a decomposition $\Pi = \Xi_0 \oplus Z_1$, where $\Xi_0 = \mathbb{Z}\{[pt]\}$, and the projection map $\Pi \to \Xi_0$ is $\Delta(0)$, which is also equal to taking the (topological) Euler characteristic χ. It follows from the proof of the decomposition ([McM89, Lemma 8]) that the kernel Z_1 of $\Delta(0)$ (equiv. χ) is generated by $[P] - 1$ (where $P \in \mathscr{P} \setminus \{\emptyset\}$). Moreover, one can show that Z_1 is nilpotent, torsion-free, divisible ([McM89, Lemma 13, 16, 17]). Hence, for $P \in \mathscr{P} \setminus \{\emptyset\}$ one can define the logarithm of $[P]$, denoted $\log[P]$ or $\mathfrak{p}$, satisfying the usual rule $\log([P_1] \cdot [P_2]) = \log[P_1] + \log[P_2]$. The component Ξ_r ($r > 0$) is then submodule of Π generated by $\mathfrak{p}^r$ for $P \in \mathscr{P} \setminus \{\emptyset\}$ (in fact any r-fold product of logarithms). Lastly, we have $\Xi_d \simeq \mathbb{R}$ by the volume form, so that $\mathrm{Vol}(P) = \mathrm{Vol}(\mathfrak{p}^d)$ or more generally $MV(P_1, \ldots, P_d) = \mathrm{Vol}(\mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_d)$.

Given a (simple) polytope $P \subset \mathbb{R}^d$, a Hodge theory is then developed on $\Pi(P)$, which is a graded subring of Π generated by Minkowski summands of P. An alternate formulation of $\Pi(P)$ is that it is generated by $\mathfrak{q}$ where Q is strongly (combinatorially) equivalent to P (in the sense that P and Q are Minkowski summands of each other). The polytopes equivalent to P, denoted $K(P)$, form a convex cone with non-empty interior in the sense that $Q_1, Q_2 \in K(P) \implies \lambda_1 Q_1 + \lambda_2 Q_2 \in K(P)$ (equiv. $\{\mathfrak{q}\}_{Q \text{ equiv to } P}$ form a convex cone in $\Pi(P)$). In this formulation, $\Xi_r(P)$ is generated by r-fold products of logarithms of polytopes equivalent to Q. We tensor $\Xi_0(P) \simeq \mathbb{Z}$ with $\mathbb{R}$ whenever necessary (i.e. when treating $\Pi(P)$ as a finite graded $\mathbb{R}$-algebra).

The first main result is that $\Pi(P)$ is the "cohomology ring" of P in the sense that $\Xi_r(P) = h_r(P)$ ([McM93, Theorem 6.1]) satisfying Poincaré duality ([McM93, Theorem 5.2]). Moreover, $\Pi(P)$

satisfies a hard Lefschetz property (LD) where an element of $K(P)$ takes the role of the ample class in classical Hodge theory ([McM93, Theorem 7.3]). The Hodge-Riemann(-Minkowski) relation (HRM) ([McM93, Theorem 8.2]) is not needed for the g-conjecture, but it is an ingredient of the proof in that HRM$(d-1)$ implies LD(d) ([McM93, Lemma 8.3]). In summary we have:

Theorem 3.3. Let $P \subset \mathbb{R}^d$ be a simple polytope, and $\mathfrak{p} = \log[P] \in \Pi(P)$. Then $\dim_{\mathbb{R}} \Xi_r(P) = h_r(P)$, and

P. $\Xi_r(P) \times \Xi_{d-r}(P) \to \Xi_d(P) \overset{\text{vol}}{\simeq} \mathbb{R}$ is a perfect pairing,

LD. $\mathfrak{p}^{d-2r} : \Xi_r(P) \xrightarrow{\sim} \Xi_{d-r}(P)$ is an isomorphism, and

HR. $\Xi_r(P) \times \Xi_r(P) \to \mathbb{R}$ via $(x, y) \mapsto (-1)^r \operatorname{vol}(x\mathfrak{p}^{d-2r}y)$ is positive definite on the primitive classes $\ker(\mathfrak{p}^{d-2r+1} : \Xi_r(P) \to \Xi_{d-r+1}(P))$.

Remark 3.4. [McM93, Theorem 14.1] The ring $\Pi(P)$ morally is really a Chow ring in the sense that $\Pi(P) \simeq R(\Sigma_P)$ where $R(\Sigma_P)$ is the usual face ring of the simplicial polytope P°. That is, indexing the primitive rays of Σ_P as $\{u_1, \dots, u_n\}$, we have

$$R(\Sigma_P) = \mathbb{R}[x_1, \dots, x_n]/I_{SR} + J$$

where $I_{SR} = \langle x^S \mid S \subset [n] \text{ not a cone in } \Sigma \rangle$ and $J = \langle \sum_{i=0}^n \langle e_k \cdot u_i \rangle x_i \mid k = 1, \dots, d \rangle$.

References

[BL81] Louis J. Billera and Carl W. Lee, *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes*, J. Combin. Theory Ser. A **31** (1981), no. 3, 237–255.

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[Huh16] June Huh, *Tropical geometry of matroids*, Current Developments in mathematics **2016** (2016), 1–46.

[McM89] Peter McMullen, *The polytope algebra*, Adv. Math. **78** (1989), no. 1, 76–130.

[McM93] ———, *On simple polytopes*, Invent. Math. **113** (1993), no. 2, 419–444.

[Zie08] Günter M. Ziegler, *Nonrational configurations, polytopes, and surfaces*, Math. Intelligencer **30** (2008), no. 3, 36–42.