TROPICALIZATION AND STABLE REDUCTION OF CURVES

CHRISTOPHER EUR

Abstract. The stable reduction theorem implies that $\mathcal{M}_g$, moduli space of curves of genus g, has a compactification to $\overline{\mathcal{M}_g}$, the Deligne-Mumford stable compactification. The space $\overline{\mathcal{M}_g}$ has a stratification where each stratum corresponds to a combinatorial type of the stable reduction. The combinatorial data of this stratification can be encoded as the moduli of tropical curves. Here we survey a relationship between these tropical curves associated to stable reductions and tropicalizations of curves.

Contents

Notation. Throughout this paper, let $\Bbbk$ denote an algebraically closed field. For a scheme X, we denote by $X(A) := \mathrm{Mor}_{Sch}(\operatorname{Spec} A, X)$ the A-valued points of X. A *curve* over $\Bbbk$ is a pure 1-dimensional reduced, separated, and finite type $\Bbbk$-scheme.

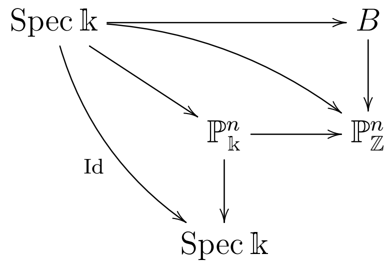
1. The moduli space $\overline{\mathcal{M}_g}$

1.1. What is a moduli space?

Informally, a **moduli space** is a geometric space whose points correspond to algebro-geometric objects of specified kind. For example, the space $\mathbb{P}^n_{\mathbb{C}}$ is a "moduli space of lines through the origin in $(n+1)$-space over $\mathbb{C}$," since the classical points of $\mathbb{P}^n_{\mathbb{C}}$ corresponds to lines in $\mathbb{C}^{n+1}$ through the origin.

To consider the notion of moduli space on a more scheme-theoretic and functorial framework, we consider the $\mathbb{P}^n_{\mathbb{Z}}$ as an example. The space $\mathbb{P}^n_{\mathbb{Z}}$ is a "moduli space of lines through the origin in $(n+1)$-space" in the sense that for any field $\Bbbk$, the $\Bbbk$-valued points $\mathbb{P}^n_{\mathbb{Z}}(\Bbbk)$ are the classical points of $\mathbb{P}^n_{\Bbbk}$. Indeed, this functor of points approach allows one to replace the base scheme $\operatorname{Spec} \Bbbk$ by any scheme B. For example, if B is a $\Bbbk$-scheme, then a point of $\mathbb{P}^n_{\mathbb{Z}}(B)$

is a map $B \to \mathbb{P}^n_{\mathbb{Z}}$ so that the closed points $B(\Bbbk)$ parameterize a family of lines through the origin of $\Bbbk^{n+1}$:



We thus arrive at the following definition:

Definition 1.1.1. A **moduli problem** consists of two data:

(1) What it means to have a *family* of desired objects over a base scheme B, giving rise to a functor $\mathsf{F} : \mathsf{Sch} \to \mathsf{Sets}$ where $\mathsf{F}(B)$ is the set of all families of the desired kind of objects over a base scheme B, and
(2) a notion $\sim$ of when two elements of $\mathsf{F}(B)$ are to be considered equivalent

from which we obtain a functor $\mathsf{M} : \mathsf{Sch} \to \mathsf{Sets}$ by $\mathsf{M}(\cdot) := \mathsf{F}(\cdot)/\sim$.

Definition 1.1.2. A scheme $\mathcal{M}$ is a **fine moduli space** for a moduli problem if it represents the functor $\mathsf{M}(\cdot)$ by $\mathcal{M}(\cdot) := \mathrm{Mor}_{Sch}(\cdot, \mathcal{M})$. As a weaker notion, $\mathcal{M}$ is a **coarse moduli space** if $\mathcal{M}(\cdot)$ admits a natural transformation $\mathsf{M}(\cdot) \to \mathcal{M}(\cdot)$ for which $\mathsf{M}(\mathrm{Spec}\,\Bbbk) \to \mathcal{M}(\Bbbk)$ is a bijection, and is the universal object with this property.

Example 1.1.3. The space $\mathbb{P}^n_{\mathbb{Z}}$ is the coarse moduli space of lines through the origin in $(n+1)$-space. To be a fine moduli space, one need generalize appropriate the notion of "lines through the origin."

Remark 1.1.4. In what follows, we define the moduli problem and the moduli space $\mathcal{M}$ *over* $\mathbb{C}$ to make our discussions easier (and due to the author's limited knowledge). However, just as $\mathbb{P}^n_{\mathbb{C}}$ could be replaced with $\mathbb{P}^n_{\mathbb{Z}}$ to provide full generality, the reader may regard $\mathcal{M}$ as a coarse moduli space as defined above to a suitable generality. (The author welcomes any comments and/or criticisms regarding this informality).

1.2. The spaces $\mathcal{M}_g, \overline{\mathcal{M}_g}$ and (semi)stable curves.

Definition/Theorem 1.2.1. The set of smooth, complete, and connected curve of genus $g \geq 2$ over $\mathbb{C}$, denoted $\mathcal{M}_g$, is a quasi-projective $\mathbb{C}$-variety of dimension $3g - 3$.

Notation. We will assume that the genus g is always ≥ 2.

That $\mathcal{M}_g$ is only quasi-projective calls for a compactification. However, naively taking its closure in the ambient projective space would not be correct, since we would like the compactification $\overline{\mathcal{M}_g}$ itself to be a moduli space. One desire is to have the boundary points $\overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ correspond to complete and connected (but necessarily singular) curves of genus g. Deligne and Mumford showed that the suitable singular curves at the boundary are **stable curves**, defined as follows:

Definition 1.2.2. A **stable curve** C is a complete and connected curve that has only nodes as singularities and has a finite automorphism group.

A smooth curve of genus ≥ 2 is indeed stable. A singular stable curve C can be thought of as finitely many curves (irreducible components) with only nodes as singularities connected together through normal crossings. With this view, since $\dim \mathrm{Aut}(\mathbb{P}^1) = 3$ whereas $\dim \mathrm{Aut}(C) \leq 1$ for $g(C) \geq 1$ (in fact 0 for $g(C) \geq 2$), we get the following equivalent definition of stable curves:

Proposition 1.2.3. A genus $g \geq 2$ curve C with only nodal singularities is stable iff each of its rational components intersects other components at least 3 times.

We now come to the main theorems of Deligne and Mumford on the stable compactification of mooduli space of curves:

Theorem 1.2.4 (Deligne-Mumford compactification)**.** The space of stable curves over $\mathbb{C}$ of genus $g \geq 2$, denoted $\overline{\mathcal{M}_g}$, is a projective variety, containing $\mathcal{M}_g$ as an open subset.

Theorem 1.2.5 (Stable reduction)**.** Let B be a smooth curve, and 0 a closed point of B, and $B^* := B \setminus \{0\}$. Let $X \to B^*$ be a flat family of stable curves of genus $g \geq 2$. Then there exists a branched cover $B' \to B$ totally ramified over 0 and a family $X' \to B'$ of stable curves extending the fiber product $X \times_{B^*} B'$. In diagram form:

$$\begin{array}{ccccc} X' & \longrightarrow & X \times_{B^*} B' & \longrightarrow & X \\ & \overset{\exists}{\searrow} & \downarrow & & \downarrow \\ & & B' & \xrightarrow{\exists} & B \end{array}$$

Morally, the stable reduction theorem means that given a 1-dimensional parameter of stable curves that limits to a non-stable singular curve, after a small base change the limit curve can be made to a stable curve. The process of finding $X' \to B'$ from $X \to B$ is called **stable reduction**.

Remark 1.2.6. In the Theorem 1.2.5, the space X' may not be smooth. One way to remedy this is to weaken the condition on the allowable curves: a complete and connected curve C of genus ≥ 2 is **semistable** if each of its rational components meets other components at least 2 times (instead of 3 times). The statement of Theorem 1.2.5 holds verbatim when the word "stable" is replaced with "semistable." The resulting X' (as a *semi*stable reduction) is now guaranteed smooth.

1.3. Stratification of $\overline{\mathcal{M}_g}$ and dual graphs.

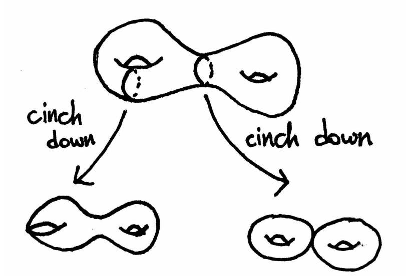
Definition 1.3.1. A space X has a **stratification** $\{U_i\}_{i \in I}$ if $X = \bigcup_i U_i$, the subsets U_i's are locally closed and disjoint, and I is a poset such that $U_i = \bigcup_{j \leq i} U_j$.

Example 1.3.2. The projective space $\mathbb{P}^n_{\Bbbk}$, Grassmannians $\mathrm{Gr}_{\Bbbk}(m, n)$, and more generally flag varieties have an affine stratification (stratification whose U_i's are $\simeq \mathbb{A}^{n_i}$).

We now describe the stratification of $\overline{\mathcal{M}_g}$ for the case $g = 2$ in an informal manner by appealing to Riemann surface pictures.

A smooth, complete, connected curve over $\mathbb{C}$ is a (compact) Riemann surface. Thus, an element of $\mathcal{M}_2$ is a Riemann surface of genus 2, whose real picture is:

We then consider a 1-dimensional family of smooth genus 2 curves limiting to a singular curve as follows: take a loop around an handle on the Riemann surface, so that the 1-dimensional parameter is the size of the loop, and as the loop shrinks down to size zero, the family limits to a singular Riemann surface. The figure below shows two such families, limiting to two different singular Riemann surfaces when the loops are cinched down to a point:



We can continue this process and obtain the following diagram of Riemann surfaces, with a corresponding diagram of complex curves (the first two of the figure below)

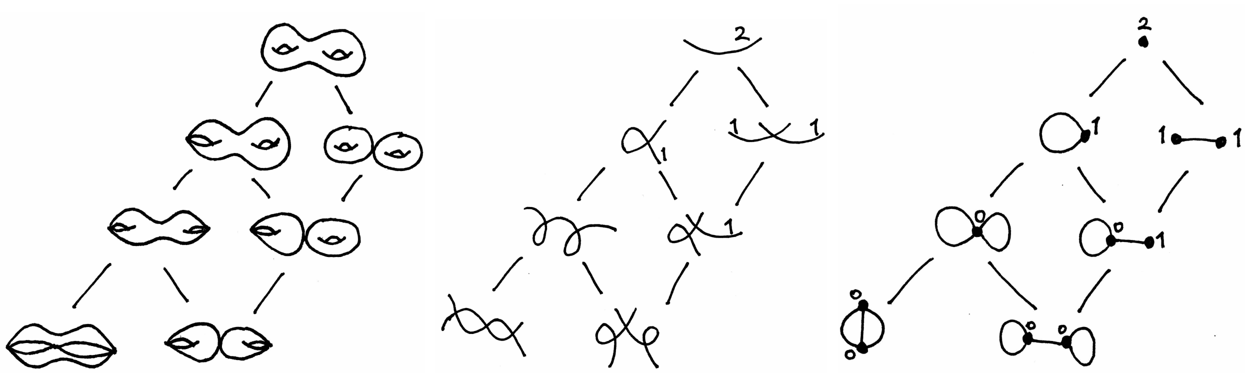


FIGURE 1. Illustrations of the stratification of $\overline{\mathcal{M}_2}$

It is a fact that $\overline{\mathcal{M}_2}$ (and $\overline{\mathcal{M}_g}$ in general) has a stratification as drawn in the diagrams above: First, $\overline{\mathcal{M}_2}$ has a (dense) open cell $\mathcal{M}_2$ whose points correspond to smooth curves of genus 2 (the top figure in the diagram), and its complement $\overline{\mathcal{M}_2} \setminus \mathcal{M}_2$ consists of points corresponding to the singular curves (union of all the figures below). The boundary $\overline{\mathcal{M}_2} \setminus \mathcal{M}_2$ itself has two open cells, corresponding to the stable curves that falls into one of two types of singular curves in the second row of the diagram. And the pattern continues.

The combinatorial types of stable curves can be encoded as follows:

Definition 1.3.3. Let C be a stable curve over $\Bbbk$. The **dual graph** of C is a finite graph with vertex weights Γ_C whose vertices are irreducible components of C with weights the genus of the component, and an edge between two vertices for each intersection of the two corresponding components.

Example 1.3.4. The dual graph diagram for $\overline{\mathcal{M}_2}$ is the third figure in Figure 1. We discuss the combinatorics of these dual graphs in the next section.

2. $\overline{\mathcal{M}_g}$ AND THE MODULI SPACE $\overline{\mathcal{M}_g}^{\text{trop}}$

2.1. **Moduli of tropical curves.**

Definition 2.1.1. A **tropical curve** Γ is a data of a graph $G = (V, E)$ with vertex weights $w : V(G) \to \mathbb{Z}$ and edge lengths $\ell : V(E) \to \mathbb{R}_{>0}$. Denoting by $ws(G)$ the sum of the weights of the vertices, define **genus of** Γ as $g(\Gamma) = |E| - |V| + 1 + ws(G)$.

Remark 2.1.2. Usually, an edge is allowed to have ∞ length if it is incident to a univalent vertex. We will not worry about this in this paper.

Given a graph G with vertex weights, we can consider a family $\mathcal{M}_G^{\text{trop}}$ of tropical curves whose underlying graph is G by varying the lengths given to the edges.

Definition 2.1.3. Let $G = (V, E)$ be a graph with vertex weights. The space $\mathcal{M}_G^{\text{trop}}$ called **moduli of tropical curves supported over** G is defined as $\mathcal{M}_G^{\text{trop}} := \mathbb{R}_{>0}^{|E(G)|} / \operatorname{Aut}(G)$.

For example, for the dumbbell graph below, denoting the three edge lengths as a, b, c (where b is the middle edge), we have that $\mathcal{M}_G^{\text{trop}}$ is $\{(a, b, c) \in \mathbb{R}_{>0}^3\} / \sim$ where $(a, b, c) \sim (a', b', c')$ if $a = c', a' = c$.

Furthermore, in the example above, as one of the edge lengths approaches zero, we get a tropical curve supported over one of the two graphs below:

In other words, the boundary of the closure $\overline{\mathcal{M}_G^{\text{trop}}}$, whatever it is, should correspond to tropical curves supported over graphs that are **contractions** of the graph G. The rule for contracting an edge of graph G to obtain a new graph G' is as follows: (i) If contracting a loop, the vertex that the loop contracts to gains $+1$ for its weight, (ii) If contracting an edge (non-loop), erase the edge and identify the two endpoints, with the new vertex weight being the sum of the two previous ones. Note that contraction preserves the genus.

Definition/Theorem 2.1.4. The **tropical moduli space** $\overline{\mathcal{M}_g}^{\text{trop}}$ is the moduli space of tropical curves of genus g. It is constructed by gluing together the spaces $\overline{\mathcal{M}_G}^{\text{trop}}$ where G's are the dual graphs arising from stable curves of genus g.

Example 2.1.5. The third column of Figure 1 can now be interpreted as giving the data of $\overline{\mathcal{M}_2}^{\text{trop}}$. The figure on the right is a depiction of $\overline{\mathcal{M}_2}^{\text{trop}}$ as a geometric space. For a picture of $\overline{\mathcal{M}_3}^{\text{trop}}$, see [Cha11].

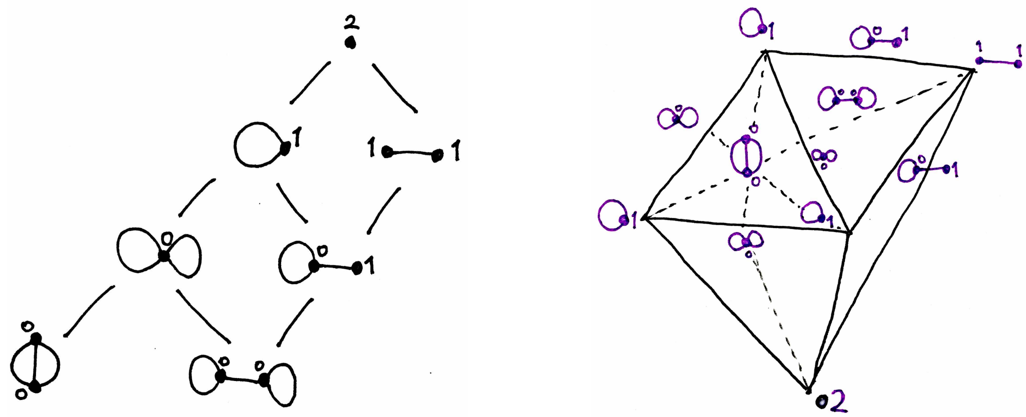


FIGURE 2. The space $\overline{\mathcal{M}_2}^{\text{trop}}$ which is supported over the dual graphs of $\overline{\mathcal{M}_g}$

Theorem 2.1.4 suggests that there is a connection between $\overline{\mathcal{M}_g}$ and $\overline{\mathcal{M}_g}^{\text{trop}}$. The stratification for both is given by the data of poset of graphs (where $G \geq G'$ if G' is given by a edge contraction of G). Calling these set of graphs $\{\Gamma\}$, we have a diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}_g} & \overset{??}{\text{- - - - -}} & \overline{\mathcal{M}_g}^{\text{trop}} \\ & \searrow \quad \swarrow & \\ & \{\Gamma\} & \end{array}$$

To explore the top connection, we need introduce the notion of Berkovich analyfication, which we do in the next subsection. We return to the relationship between $\overline{\mathcal{M}_g}$ and $\overline{\mathcal{M}_g}^{\text{trop}}$ the subsection after.

2.2. **Berkovich analyfication and tropicalization.**

We first give a cursory treatment of Berkovich analyfication, just enough to provide an idea of what it is. The main reference is [Bak07] (this is a topic the author hopes to study more and expand upon in the future).

Notation. Throughout this subsection, let K be a field with valuation $\nu : K^\times \to \mathbb{R}$.

Definition 2.2.1. A **(multiplicative) seminorm** on a ring R is a map $\|\cdot\| : R \to \mathbb{R}_{\geq 0}$ such that (i) $\|ab\| = \|a\|\|b\|$, (ii) $\|a+b\| \leq \|a\| + \|b\|$, and (iii) $\|0_R\| = 0$, $\|1_R\| = 1$.

Here (for now), we only define what Berkovich analyfication is for an affine scheme.

Definition 2.2.2. Let K be a valued field with the induced norm $|\cdot|$. If A is a K-algebra and $X = \operatorname{Spec} A$, define X^{an}, the Berkovich analyfication of X, as

$$X^{an} := \{\|\cdot\|_x : \text{ a seminorm on } A \text{ extending the norm on } K\}$$

with the weakest topology such that for any $f \in A$, the map $f : \|\cdot\|_x \mapsto \|f\|_x$ is continuous.

Remark 2.2.3. X^{an} is locally compact and Hausdorff. Berkovich's theorem on types of points for $(\mathbb{A}^1_K)^{an}$ allows one to visualize it as and infinite tree, whose branches also has a dense set of points from which infinite trees branch out.

Note 2.2.4. The K-valued points $X(K)$ naturally embed into X^{an} as follows: For $x \in X(K)$, define $\|\cdot\|_x : f \mapsto |f(x)| \; \forall f \in A$.

We now describe Payne's result on tropicalization and analyfication following .

Recall that given X, an affine subvariety over K of a m-torus $T^m(K)$, its tropicalization $\mathrm{trop}(X) = \mathrm{cl}\{(\nu(y_1), \nu(y_2), \ldots, \nu(y_n)) \in \mathbb{R}^m : (y_1, \ldots, y_n) \in X(K)\}$ (where the closure is the usual Euclidean closure). Extending the valuation on $K^\times$ to $\nu(0) := \infty$, and defining $\mathbf{R} := \mathbb{R} \cup \{\infty\}$ (with the topology homeomorphic to the interval $(0, 1]$), we extend this tropicalization to $\mathrm{trop}(X) \subset \mathbf{R}^m$ where X is now an affine K-variety in $\mathbb{A}^m_K$. For an abstract affine K-variety X with coordinate ring $K[X]$, the tropicalization $\mathrm{trop}(X)$ depends on the embedding $\iota : X \hookrightarrow \mathbb{A}^{m_\iota}$. So, denote by $\mathrm{trop}(X, \iota)$ the tropicalization of X given an embedding ι.

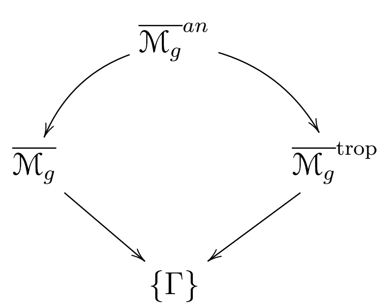
For each ι, there is a natural map $\pi_\iota : X^{an} \to \mathrm{trop}(X, \iota)$ as follows. Let $y_1, \ldots, y_m \in K[X]$ be the coordinate functions of the embedding. Then define the map $\pi_\iota : X^{an} \to \mathbf{R}^m$ by $\|\cdot\|_x \mapsto (-\log \|y_1\|_x, \ldots, -\log \|y_m\|_x)$. Indeed, if $x \in X(K)$, then $-\log \|y_i\| x = -\log |x_i| = -\log(\exp(-\nu(x_i))) = \nu(x_i)$, so that the Euclidean closure of $\mathrm{Im}(\pi_\iota)$ is $\mathrm{trop}(X, \iota)$.

Theorem 2.2.5. [Pay09] Let X be an affine K-variety. The map $\pi_\iota : X^{an} \twoheadrightarrow \mathrm{trop}(X, \iota)$ is surjective, and moreover, the induced map $\varprojlim_\iota \pi_\iota : X^{an} \xrightarrow{\sim} \varprojlim_\iota \mathrm{trop}(X, \iota)$ is a homeomorphism. Moreover, this result can be extended to the case where X is a quasi-projective K-variety.

In the next subsection, we finally come to the discussion of what these tools have to do with stable reduction of curves.

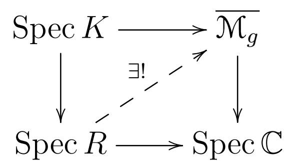
2.3. Connecting $\overline{\mathcal{M}_g}$ to $\overline{\mathcal{M}_g}^{\mathrm{trop}}$.

$\overline{\mathcal{M}_g}^{an}$, the Berkovich analyfication of $\overline{\mathcal{M}_G}$, provides the link between $\overline{\mathcal{M}_g}$ and $\overline{\mathcal{M}_g}^{\mathrm{trop}}$. We now describe the two new arrows in the diagram below:



Note 2.3.1 (Left arrow). *Fact:* Points of $\overline{\mathcal{M}_g}^{an}$ correspond to a valued field extension $K \supset \mathbb{C}$ (extending trivial valuation on $\mathbb{C}$) and a map $\mathrm{Spec}\, K \to \overline{\mathcal{M}_g}^{an}$. As a point of $\overline{\mathcal{M}_g}$ is a field extension $K \supset \mathbb{C}$ and a map $\mathrm{Spec}\, K \to \overline{\mathcal{M}_g}$, we indeed get a map $\overline{\mathcal{M}_g}^{an} \to \overline{\mathcal{M}_g}$.

Note 2.3.2 (Right arrow)**.** Theorem 1.2.4 in particular implies that $\overline{\mathcal{M}_g}$ is proper over $\mathbb{C}$. Now, suppose $K \supset \mathbb{C}$ is a valued field extension (extending the trivial valuation on $\mathbb{C}$), so that its valuation ring R contains $\mathbb{C}$. Suppose we have a smooth curve of genus g over K, i.e. a K-valued point of $\mathcal{M}_g$. We have a diagram below, and by valuative criterion of properness, there exists a unique map $\operatorname{Spec} R \to \overline{\mathcal{M}_g}$



in summary, given a smooth K-curve X where $K \supset \mathbb{C}$ is a valued field extension, there exists a (unique) scheme $\mathcal{X}$ over R, called the **stable model of** X, such that the fiber over the generic point is X and the fiber over the closed point is a stable curve $\mathcal{X}_k$ over k (the residue field of R).

Thus, given a point on $\overline{\mathcal{M}_g}^{an}$, i.e. a curve X over $K \supset \mathbb{C}$ a valued field extension, take the stable model $\mathcal{X}$ and consider the dual graph $\Gamma_{\mathcal{X}_k}$ of $\mathcal{X}_k$. The edge lengths of $\Gamma_{\mathcal{X}_k}$ are given as follows: a locally around a normal crossing in $\mathcal{X}_k$, on $\mathcal{X}$ (before specializing) the local equation is $xy = f$ for $f \in \operatorname{Spec} R$. Now, the edge length of $\Gamma_{\mathcal{X}_k}$ corresponding to the normal crossing is $\nu(f)$.

2.4. **Computing the (semi)stable model.**

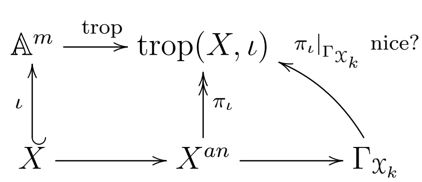
While there is an algorithm for carrying out the stable reduction (outlined in [HM98, §3.C]), it involves several steps of blowing-up, normalization, and blowing-down that makes it computationally rather difficult. To the author's knowledge, it is not even easy to tell what combinatorial type the dual curve of the resulting stable curve is until the lengthy algorithm has been carried out till the end.

We survey here how tropical geometry can pave a way to a computationally efficient way to compute the stable reduction. The main fact is as follows:

Theorem 2.4.1. [Bak07, §5] Let X be a smooth curve over a valued field K, and let $\mathcal{X}$ be a stable model. Then there exists a deformation retract $r : X^{an} \to \Gamma_{\mathcal{X}_k}$ where $\Gamma_{\mathcal{X}_k}$ is the dual graph of the stable curve $\mathcal{X}_k$. The space $\Gamma_{\mathcal{X}_k}$ as a subspace of X^{an} is called the **Berkovich skeleton** of X.

Remark 2.4.2. The above theorem is in fact stated for *semi*stable models of X in [Bak07, §5.1]. It is the author's impression that it also holds for stable models. Moreover, there seems to be several ways to approach what "Berkovich skeleton" is (see [BPR11, §1.2]), and the notion of Berkovich skeleton is a complex one that the author does not understand well. The author apologizes and welcomes criticisms for any errors introduced here.

Given the theorem above, and recalling the relationship between analyfication and tropicalization, we have the following diagram:

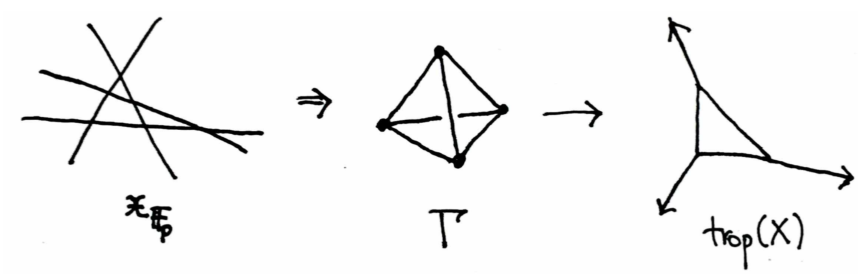


We conclude with a discussion of when the map $\pi_\iota|_{\Gamma_{\mathfrak{X}_k}}$ is "nice" in the sense that trop(X,ι) faithfully represents the dual graph $\Gamma_{\mathfrak{X}_k}$.

Example 2.4.3. Consider a plane curve X over $\mathbb{C}_p$ (for $p \geq 5$) given by

$$f(x,y,z) = x^3y - x^2y^2 - 2xy^3 - 3x^2yz + 2xyz^2 - pz^4 = 0$$

One can check by Jacobian condition that X is a smooth curve over $\mathbb{C}_p$. Moreover, all the coefficients have non-negative valuation, and hence we may consider $\mathfrak{X}$, a scheme over the valuation ring of $\mathbb{C}_p$ given by the same polynomial. $\mathfrak{X}$ is in fact a stable model since $\mathfrak{X}_{\mathbb{F}_p}$ is given by an equation $\overline{f}(x,y,z) = xy(x+y-z)(x-2y-2z)$, which is four lines in general position. Hence, the dual graph Γ is the $K4$-graph, but the map to trop(X) destroys some of this information:



On the other hand, a curve over $\mathbb{C}\{\{t\}\}$ defined by $f = t^4(x^4+y^4+z^4) + t^2(x^3y + x^3z + xy^3 + y^3z + xz^3 + yz^3) + t(x^2y^2 + x^2z^2 + y^2z^2) + xyz(x+y+z)$ has the same stable reduction type $xyz(x+y+z)$, but trop$(V(f))$ has $K4$ as its bounded part (see [BPR16, 5.29])

The crux of the problem is to find the right embedding ι such that the bounded part of trop(X,ι) is homeomorphic (even better: isometric) to $\Gamma_{\mathfrak{X}_k}$. Here is one criteria for testing whether trop(X,ι) is a faithful representation of $\Gamma_{\mathfrak{X}_k}$:

Theorem 2.4.4. [BPR16, 5.28] Suppose X is a smooth, complete, projective K-curve of genus $g(X) \geq 1$ and $\Gamma_{\mathfrak{X}_k}$ its Berkovich skeleton (dual graph of the stable reduction).

(1) If X is a plane curve given by $f \in K[x,y]$ whose Newton complex is a unimodular triangulation, then $\mathfrak{X}_k$ is a totally degenerate reduction and trop : $\Gamma_{\mathfrak{X}_k} \to$ trop(X) is an isometry onto its image.

(2) More generally, if all vertices of trop(X) are trivalent, all edges of trop(X) have multiplicity 1, the graph $\Gamma_{\mathfrak{X}_k}$ has no vertex of valence 1, and $\dim H_1(\Gamma_{\mathfrak{X}_k},\mathbb{R}) = \dim H_1(\text{trop}(X),\mathbb{R})$, then trop : $\Gamma_{\mathfrak{X}_k} \to$ trop(X) is an isometry onto its image.

Thus, we have trop(X) as a faithful representation of $\Gamma_{\mathfrak{X}_k}$, we can ask for an embedding $\iota : X \hookrightarrow \mathbb{P}^n$ such that the Newton complex has a unimodular triangulation.

References

[Abr13] **Abramovich, D.**. *Moduli of algebraic and tropical curves.* http://arxiv.org/abs/1301.0474

[Bak07] **Baker, M.**. *An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves.* Lecture notes written for Arizona Winter School 2007, Southwest Center for Arithmetic Geometry. http://people.math.gatech.edu/~mbaker/pdf/aws07mb_v4.pdf

[BPR11] **Baker, Payne, & Rabinoff**. *On the structure of nonarchimedean analytic curves.* http://arxiv.org/abs/1404.0279

[BPR16] **Baker, Payne, & Rabinoff**. *Nonarchimedean geometry, tropicalization, and metrics on curves.* Algebraic Geometry 3 (2016): 63-105.

[Cha11] **Chan, M.**. *Combinatorics of the tropical Torelli map.* http://arxiv.org/abs/1012.4539

[HM98] **Harris & Morrison**. *Moduli Space of Curves.* Springer-Verlag, GTM 187, (1998).

[MS15] **Maclagan & Sturmfels**. *Introduction to Tropical Geometry.* AMS GSM 161, (2015).

[Pay09] **Payne, S.**. *Analytification is the limit of all tropicalizations.* Math. Res. Lett. 19 (2009): 543-556.

[RSS] **Ren, Sam, & Sturmfels**. *Tropicalization of classical moduli spaces.* http://arxiv.org/abs/1303.1132