On Newton-Okounkov bodies

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Abstract

We summarize recent developments regarding Newton-Okounkov bodies which can be viewed as a generalization of the polytope of projective toric variety or the Newton polytope a polynomial. We start with a discussion of formal properties of integral semigroups and valuations of graded algebras, the general setting in which Newton-Okounkov bodies can be defined. We then consider some applications to intersection theory in algebraic geometry. Lastly, we consider its relation to SAGBI bases and toric degenerations.

Newton-Okounkov (NO) bodies were introduced in passing in the works of Okounkov ([Ok96], [Ok03]), and were subsequently studied more systematically in [LM08] and [KKh08]. Several subsequence works ([KKh12], [And13], [Kav15]) have emerged since, and NO bodies continue to be an active area of research.

In Section §1, we give some general properties of integral semigroups and valuations of graded algebras, as this is the general setting in which Newton-Okounkov bodies can be defined. In Section §2, we show how Newton-Okounkov bodies naturally arise in the setting of algebraic varieties, and apply the tools developed in §1 to intersection theory following [LM08] and [KKh12]; these applications include generalizations of Fujita approximation, Brunn-Minkowski inequality, and the Bernstein-Kushnirenko theorem. In Section §3 we discuss the recent connection of NO bodies to SAGBI bases and toric degenerations.

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1 Generalities on semigroups and valuations of graded rings

1.1 Integral semigroups

A motivational example to keep in mind throughout this article the Hilbert function of a projective variety, particularly that of a normal projective toric variety.

Example 1.1.1. Let $X \subset \mathbb{P}^n_k$ be a d-dimensional projective scheme over a field k given by a graded ideal I of the standard graded polynomial ring $R = k[x_0, \ldots, x_n]$. The Hilbert function $H_X(t) := \dim_k(R/I)_t$ coincides with the Hilbert polynomial $P_X(t) := \chi(\mathscr{O}_X(t))$ of degree d for $t >> 0$, so that $\lim_{t\to\infty} H_X(t)/t^d$ is well-defined. The value of this limit time $d!$ is the degree of the embedded scheme $X \subset \mathbb{P}^n$.

We now consider this for projectively normal toric varieties. Let Q be a full-dimensional lattice polytope in $\mathbb{Z}^n$ whose lattice points are $A = \{a_1, \ldots, a_m\} \subset \mathbb{Z}^n$, and let $\hat{C}_Q := \mathrm{Cone}((a_i, 1) \in \mathbb{R}^n \times \mathbb{R}) \cap (\mathbb{Z}^n \times \mathbb{Z})$ be the rational polyhedral cone over Q. Assume that Q is **normal** (or IDP)—i.e. all points of $\hat{C}_Q$ are $\mathbb{Z}$-linear combinations of $\{(a_i, 1)\}$. Consider the projective toric variety X_A defined by closure of the image of the map $(\mathbb{C}^*)^n \times (\mathbb{C}^*) \to \mathbb{P}^{m-1}$, $(\underline{t}, s) \mapsto [\underline{t}^{a_1} s : \cdots : \underline{t}^{a_m} s]$. The graded coordinate ring of X_A is the graded semigroup algebra $\mathbb{C}[\hat{C}_Q]$, where the degree of $\chi^{(a',l)} \in \mathbb{C}[\hat{C}_Q]$ for $(a', l) \in \hat{C}_Q \subset \mathbb{Z}^n \times \mathbb{Z}$ is l. In other words, the Hilbert function is $H_{X_A}(t) = \#|\hat{C}_Q \cap (\mathbb{Z}^n \times \{t\})|$. Standard Ehrhart theory then tells us that $\lim_{t\to\infty} H_{X_A}(t)/t^n = \mathrm{vol}(Q)$; in other words, the degree of the projectively normal toric variety is given by the volume of its defining polytope.

Remark 1.1.2. The statements for toric varieties above hold without change when Q is very ample (in fact, normal implies very ample). For discussion of normality and very ampleness of a lattice polytope, see [CLS11, Ch. 2].

We now generalize the above picture to a general setting. By a **(integral) semigroup** we mean commutative semigroup $S \subset \mathbb{Z}^n$. Let the **linear space** $L(S)$, **group** $G(S)$, and **cone** $C(S)$ **of** S to be $L(S) = \mathbb{R}S \subset \mathbb{R}^n$, $G(S) = \mathbb{Z}S \subset \mathbb{Z}^n$, and $C(S) = \overline{\{\sum_i c_i s_i \in \mathbb{R}^n \mid c_i \geq 0,\ s_i \in S\}}$.

For a lattice $N \simeq \mathbb{Z}^n$, denote by its dual $N^\vee := \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and let $\langle \cdot, \cdot \rangle : N^\vee \times N \to \mathbb{Z}$ be the usual pairing. A primitive element $m \in N^\vee$ defines a half-space $M := \{u \in N : \langle m, u \rangle \geq 0\}$ and a map $\pi_M := \langle m, \cdot \rangle$ such that $\pi_M(N) = \mathbb{Z}$ and $\ker \pi_M = \partial M$ is the boundary of M.

Definition 1.1.3. Let S be a semigroup and M a rational half-space in $L(S)$ given by a primitive element of $(L(S) \cap \mathbb{Z}^n)^\vee$. We say that (S, M) is **admissible** if $S \subset M$, and **strongly admissible** if $C(S)$ is strongly convex.

Definition 1.1.4. For an admissible pair (S, M), define its **Newton-Okounkov body** (NO body) $\Delta(S, M)$ to be

$$\Delta(S, M) := C(S) \cap \pi_M^{-1}(m) \quad \text{where } m = [\mathbb{Z} : \pi_M(G(S))].$$

Note by construction that $\Delta(S, M)$ is compact iff (S, M) is strongly admissible. When S is the semigroup of the cone $\hat{C}_Q$ over a normal (or very ample) lattice polytope $Q \subset \mathbb{Z}^n$, and M is the half-space $\mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$, the NO body of (M, S) is Q.

For an admissible pair (S, M) and $k \in \mathbb{Z}_{\geq 0}$, denote by $S_k := C(S) \cap \pi_M^{-1}(k)$ and call $H(k) := \#|S_k|$ the Hilbert function of (S, M). The following is the generalization of observations made in Example 1.1.1.

Theorem 1.1.5. [KKh12, Theorem 1.19] Let (S, M) be a strongly admissible pair, and assume that S_k is finite for all $k \in \mathbb{Z}_{\geq 0}$. Let $q := \dim \partial M = \dim L(S) - 1$, $m := [\mathbb{Z} : \pi_M(G(S))]$, and $c := [\partial M : G(S) \cap \partial M]$. Then

$$\lim_{k\to\infty} \frac{H(mk)}{k^q} = \frac{1}{c} \mathrm{vol}(\Delta(S, M))$$

where the volume measure on $L(S) \cap \pi_M^{-1}(m)$ is normalized so that a standard integral q-simplex has volume $1/q!$.

Sketch of proof. First define the **regularization** of a semigroup S as $\operatorname{Reg}(S) := G(S) \cap C(S)$, which approximates S well in the following sense ([KKh12, Theorem 1.6]): there exists a constant $N > 0$ such that $\operatorname{Reg}(S) \cap \{s \in \mathbb{Z}^n : |s| > N\} \subset S$. Then a series of reductions, outlined in the proof of [KKh12, Theorem 1.14], gives a generalization of the desired result above that goes back to Minkowski. $\square$

Note that when $G(S) = \mathbb{Z}^n$, we have $m = 1$ and $c = 1$.

Example 1.1.6 (Sanity check)**.** Suppose $S = \mathbb{Z}_{\geq 0}^{n+1}$ and $\pi_M : \mathbb{R}^n \to \mathbb{R}$ via $\pi_M(\vec{a}) = \sum_{i=0}^n a_i$ (i.e. M is half-space defined by the all-1-vector as the normal vector). Then the Newton-Okounkov body (S, M) is the n-simplex $\operatorname{Conv}(e_0, \ldots, e_n)$ which has volume $1/n!$. On the other hand, $H(k) = \binom{n+k}{n}$ so that $\lim_{k\to\infty} H(k)/k^n = 1/n!$.

By a **strongly non-negative** semigroup S, we mean a strongly admissible pair (S, M) where $S \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$ and $M = \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$. In almost all cases, we'll be working with strongly non-negative semigroups. For S strongly non-negative, define $\hat{S}_p$ to be the subsemigrouop of S generated by $S \cap \pi_M^{-1}(p)$. Note that for large enough multiple p of $m := [\mathbb{Z} : \pi_M(G(S)])$, the subsemigroup $\hat{S}_p$ is nonempty.

The following theorem states that the volume of $\Delta(S)$ can be approximated through by volume of $\Delta(\hat{S}_{mk})$ for large enough k.

Theorem 1.1.7. [KKh12, Theorem 1.27] Let S be a strongly non-negative semigroup. Then for any sufficiently large multiple p of $m := [\mathbb{Z} : \pi_M(G(S))]$, we have $q := \dim \Delta(S) = \dim \Delta(\hat{S}_p)$ and $c := [\partial M, G(S) \cap \partial M] = [\partial M, G(\hat{S}_p) \cap \partial M]$. Moreover,

$$\frac{1}{c}\operatorname{vol}(\Delta(S)) = \lim_{k\to\infty}\left(\frac{1}{k^q}\lim_{t\to\infty}\frac{H_{\hat{S}_{mk}}\Big((mk)t\Big)}{t^q}\right) = \frac{1}{c}\lim_{k\to\infty}\frac{\operatorname{vol}(\Delta(\hat{S}_{mk}))}{k^q}.$$

1.2 Valuations on graded rings

Let Γ be a totally ordered Abelian group and k a (algebraically closed) field. By **prevaluation** on a k-vector space V, we mean a map $\nu : V \to \Gamma \cup \{\infty\}$ such that $\nu(u) = \infty \Leftrightarrow u = 0$, $\nu(f + g) \geq \min(\nu(f), \nu(g))$, and $\nu(cf) = \nu(f)$ for all $f, g \in V$, $c \in k$. A **valuation** on k-algebra A further satisfies $\nu(fg) = \nu(f) + \nu(g)$. A valuation is **faithful** if $\nu(A \setminus \{0\}) = \Gamma$. Denote by $V_{\geq\alpha} := \{f \in V : \nu(f) \geq \alpha\}$ for $\alpha \in \Gamma$ (and likewise $V_{>\alpha}$). For most cases, we will concern the following type of valuation.

Definition 1.2.1. Let $R_\bullet = \bigoplus_{i\geq 0} R_i$ be a ($\mathbb{Z}_{\geq 0}$-)graded k-algebra. A **graded valuation on** R is a valuation $\nu : R \to \mathbb{Z}^n \times \mathbb{Z}$ where $\nu(R_i) \in \mathbb{Z}^n \times \{i\}$ and $\mathbb{Z}^n \times \mathbb{Z}$ is totally ordered by $(\alpha, m) < (\beta, n) \iff m > n$ or $m = n$ and $\alpha <_{lex} \beta$.

In many cases of (pre)valuations, we will have that $V_{\geq\alpha}/V_{>\alpha}$ has at most dimension 1, in which case we say (V, ν) has **one-dimensional laves**. The number of values a vector space takes is its dimension in this case:

Proposition 1.2.2. Let V be a k-vector space with prevaluation ν with one-dimensional leaves. Then for any nonzero subspace $W \subset V$ we have $\dim W = \#|\nu(W \setminus \{0\})|$.

Proof. By the well-known property of valuation that $\nu(f+g) > \min(\nu(f), \nu(g)) \implies \nu(f) = \nu(g)$, one notes that $\{f_1, \ldots, f_n\} \subset V \setminus \{0\}$ are linearly independent if $\nu(f_i)$'s are distinct. Now, that the leaves are 1-dimensional implies that there exists a basis V consisting of distinct ν-values. Lastly, restricting to subspace behaves well with prevaluations. $\square$

For a graded valuation $(\nu, R_\bullet)$, define its associated semigroup $S(R) := \nu(R \setminus \{0\}) \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$. If the valuation has one-dimensional leaves, then the classical Hilbert function $H_R(t) = \dim_k R_t$ and the Hilbert function of the associated semigroup $H_{S(R)}(t) := \#|S(R) \cap (\mathbb{Z}^n \times \{t\})|$ are the same.

2 Applications to Intersection theory

Throughout this section, let k be an algebraically closed field of characteristic 0. Let X be a n-dimensional normal projective k-variety (irreducible and reduced). Many of the results here can be generalized to arbitrary complete k-varieties, using normalization and the Chow lemma, but we'll assume normal and projective for ease of exposition. For details on generalizations, see [LM08] and [KKh12].

2.1 Newton-Okounkov body of a variety

Let $K(X)$ be the field of rational functions of X. We first describe a faithful valuation on $K(X)$ as a k-algebra.

Example 2.1.1 (Gröbner valuation)**.** Let $p \in X$ be a regular point, so that $\mathscr{O}_{X,p}$ is a regular local ring, with a regular system of parameters $(u_1, \ldots, u_n)$. By Cohen structure theorem the completion of $\mathscr{O}_{X,p}$ is the power series ring $A := k[[u_1, \ldots, u_n]]$, so that $K(X)$ injects into $\operatorname{Frac} A$. Consider the Gröbner valuation ν on A where $\nu(f \in A)$ is the exponent of the minimal nonzero monomial term in f (under some monomial ordering, say $<_{lex}$). Restricting ν to $K(X)$ gives a valuation on $K(X)$. This valuation ν depends on the choice of the point p and the system of parameters $(u_1, \ldots, u_n)$.

A more geometric and way to state the above construction is as follows:

Example 2.1.2 (Parshin valuation)**.** An **admissible flag** $Y_\bullet$ on X is a flag of subvarieties

$$X = Y_0 \supset Y_1 \supset \cdots \supset Y_d$$

such that $\operatorname{codim}_X(Y_i) = i$ and the point Y_d is a regular point of X. In other words, take a regular point $p = Y_d$ and consider a maximal chain of prime ideals $(0) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d = p$ in $R := \mathscr{O}_{X,p}$. For example, if $(u_1, \ldots, u_n)$ is a system of parameters of $\mathscr{O}_{X,p}$ then take $\mathfrak{p}_i$ to be a minimal prime over $(u_1, \ldots, u_i)$. Note that by assumption $R_{\mathfrak{p}_i}/\mathfrak{p}_{i-1}$ is a regular of dimension 1 so that it is a DVR with a uniformizer π_i.

We now define a valuation $\nu_{Y_\bullet}$ on $K(X)$ associated to an admissible flag. Let $f \in K(X)$. Set $f_1 = f$. As $R_{\mathfrak{p}_1}/\mathfrak{p}_0 = R_{\mathfrak{p}_1} \subset K(X)$ is a DVR, we have that $f_1 = \hat{f}_2 \cdot \pi_1^{\nu_1(f)}$ for some integer $\nu_1(f)$ and $\hat{f}_2 \in K(X) \setminus \mathfrak{p}_1 R_{\mathfrak{p}_1}$. Writing $\hat{f}_2 = \phi/\psi$ where $\phi, \psi \in R_{\mathfrak{p}_2} \setminus \mathfrak{p}_1$, we can consider a rational function f_2 as the image of $\hat{f}_2$ in the fraction field of $R_{\mathfrak{p}_2}/\mathfrak{p}_1$. Then we can write $f_2 = \hat{f}_3 \pi_2^{\nu_2(f)}$, and so forth. Continuing this process gives us a valuation $\nu = (\nu_1, \ldots, \nu_n)$. The valuation is only dependent on the choice of the admissible flag ([LM08, §1]).

For the rest of the section, fix an admissible flag and hence fix a faithful valuation ν on $K(X)$ given by the construction above. The Grobner valuation perspective of Example 2.1.1 makes it clear that the valuation has one-dimensional leaves; for the geometric perspective of Example 2.1.2, a geometric proof of one-dimensional leaves can be found in [LM08, Lemma 1.3].

By restriction one can define a graded valuation $\widetilde{\nu}$ on any graded k-subalgebra $W_\bullet$ of $K(X)$ by $\widetilde{\nu}(w \in W_t) = (\nu(w), t) \in \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$. Divisor theory provides some natural graded k-subalgebras of $K(X)$ as follows.

For a line bundle $\mathscr{L}$ on X, denote by its **section ring**

$$R(\mathscr{L}) = \bigoplus_{i\geq 0} H^0(X, \mathscr{L}^{\otimes i}),$$

and given a subspace $L \subset H^0(X, \mathscr{L})$, define the graded ring

$$R_L := \bigoplus_{i\geq 0} L^i$$

where L^i is the image of the map $\bigotimes_{j=1}^{i} H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L}^{\otimes i})$ restricted to $\bigotimes_j L$. (Viewing elements of L as rational functions (since $H^0(X, \mathscr{L}) \subset K(X)$), one can just take products in $K(X)$).

More generally, we consider the following class of graded k-subalgebras of $K(X)$:

Definition 2.1.3. Let $L \subset K(X)$ be a finite dimensional k-vector subspace of $K(X)$, and let $R_L := \bigoplus_{i\geq 0} L^i$. A graded k-subalgebra of $K(X)$ is **of almost integral type** if it is contained in a k-subalgebra of $K(X)$ that is finite over R_L for some $L \subset K(X)$.

When $\mathscr{L}$ is very ample, and L is a subsystem of the complete linear system $|\mathscr{L}|$ giving a projective embedding $X \subset \mathbb{P}(L)$, then R_L is the graded coordinated ring of $X \subset \mathbb{P}(L)$ and $R(\mathscr{L})$ is the normalization of R_L ([Har77, II.5]). Hence, in this case, any graded k-subalgebra of $R(\mathscr{L})$ is of almost integral type. In fact, noting that $D + kE$ is very ample for any Cartier divisor D and very ample divisor E for sufficiently large k, one can conclude that $R(\mathscr{O}(D))$ is of almost integral type for any Cartier divisor D on X ([KKh12, Theorem 3.7]). In other words, any k-subalgebra of ring of sections of a line bundle is of almost integral type.

Theorem 2.1.4. [KKh12, Thoerem 2.30] (or [LM08, Lemmas 2.2,4,6,12]). Let $A \subset K(X)$ be a graded k-subalgebra of almost integral type, and let $\widetilde{\nu}$ be a valuation obtained by the construction in 2.1.2. Then the semigroup $S(A) = \nu(A \setminus \{0\})$ is a strongly non-negative semigroup, and hence its Newton-Okounkov body $\Delta(S(A))$ is compact.

For a line bundle $\mathscr{L}$, we define its **Newton-Okounkov body** $\Delta(\mathscr{L})$ to be $D(S(R(\mathscr{L})))$. More generally, we define $\Delta(A)$ to be the Newton-Okounkov body of $S(A)$ for any A, a graded k-subalgebra of almost integral type. We'll use the notation $\Delta_{Y_\bullet}$ when we wish to be explicit about which admissible flag is used to construct the valuation.

2.2 Toric examples

We work out explicitly examples of Newton-Okounkov bodies for projective toric varieties. First, we follow [LM08] in a geometrically nice case of smooth toric varieties.

Let X_Σ be a complete smooth toric variety (over $\mathbb{C}$, or any algebraically closed field of characteristic 0), given by a complete smooth fan $\Sigma \subset N \simeq \mathbb{Z}^n$. Take a maximal cone $\sigma \in \Sigma$; its primitive rays $(u_1, \ldots, u_n)$ form a basis of N. Let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the character lattice, and let $\{u_1, \ldots, u_n, u_{n+1}, \ldots, u_r\}$ be the (primitive) rays of Σ and D_i their corresponding divisors. Fix the admissible flag $Y_\bullet$ to be $Y_i = D_1 \cap \cdots \cap D_i$ or $1 \leq i \leq n$. Then as D_i's have simple-normal crossings, for $f \in \mathbb{C}(X_\Sigma)$ we have $\mathrm{div}(f) = \sum_{j=1}^{r} a_j D_j$ then $\nu(f) = (a_1, \ldots, a_n)$.

Recall that for a torus invariant (Weil) divisor $D = \sum_i a_i D_i$ on a toric variety, one can identify $H^0(X_\Sigma, \mathscr{O}(D))$ with $\bigoplus_{m\in P_D} \chi^m \subset \mathbb{C}[M]$ where $P_D \subset M$ is defined by

$$P_D = \{m \in M \mid \langle u_i, m\rangle + a_i \geq 0 \ \forall i\}.$$

since for a rational function χ^m we have $\operatorname{div}\chi^m = \sum_i \langle m, u_i\rangle D_i$.

To make this construction invariant under rational equivalence of divisors, consider a map $\varphi_D = M \to \mathbb{Z}^r$ by $m \mapsto \sum_i(\langle m, u_i\rangle + a_i)\vec{e}_i$. The image $\varphi_D(M)$ is a sublattice of $\mathbb{Z}^r$ of index 1 since Σ is smooth and complete. Moreover, $\varphi_D(M) \cap \mathbb{Z}^r_{\geq 0} = \varphi(P_D) \simeq P_D$. We show that P_D is the Newton-Okounkov body $\Delta(D)$ for any Cartier divisor D. (This generalizes [LM08, Proposition 6.1]).

Proposition 2.2.1. Let $\mathscr{L}$ be a line bundle on X_Σ, and let D be a torus invariant Cartier divisor such that $\mathscr{O}(D) = \mathscr{L}$. Then the Newton-Okounkov body $\Delta(D)$ is equal to the polytope $\varphi_D(P_D)$.

Proof. Let $\chi^m \in H^0(X, \mathscr{L})$. Then $\nu(\chi^m) = (\langle m, u_1\rangle, \ldots, \langle m, u_n\rangle)$. But as $u_1, \ldots, u_n$ is a basis of N, the map $\nu : M \xrightarrow{\text{div}} \mathbb{Z}^n \times \mathbb{Z}^{r-n} \xrightarrow{\pi} \mathbb{Z}^n$ is in fact an isomorphism. For normal toric varieties, as $P_{kD} = kP_D$ for any effective divisor D, we need only look at $k = 1$ level for the Newton-Okounkov body. □

More generally, consider a projective toric variety obtained by a finite subset $A = [a_0, \ldots, a_m]$ (considered as a $n \times (m+1)$ matrix) of a lattice $M \simeq \mathbb{Z}^n$ whose affine $\mathbb{Z}$-span is $\mathbb{Z}^n$ (when it doesn't just restrict to a sublattice). Add a row of 1's to A to get a a $(n+1) \times (m+1)$ matrix A' defining a n-dimensional projective toric variety $X_A := \overline{\{[t^{(1,a_0)} : \ldots : t^{(1,a_m)}]\}} \subset \mathbb{P}^m$. The rational function field $K(X_A)$ is the quotient field of $\mathbb{C}[(t_1/t_0)^{\pm}, \ldots, (t_n/t_0)^{\pm}]$, so that with Gröbner valuation on $K(X_A)$ we get $S(R_X)$ to be the semigroup $\mathbb{N}A'$ where R_X is the graded coordinate ring of $X_A \subset \mathbb{P}^m$. It is then easy to see that the Newton-Okounkov body is the convex hull of A.

Example 2.2.2 (Sanity check)**.** Let's consider three semigroups in $\mathbb{Z}_{\geq 0}$: $A_1 = [0, 1, 3, 4]$ (giving the rational twisted quartic), $A_2 = [0, 2, 3]$ (cuspdial cubic), and $A_3 = [0, 1, 2, 3]$ (the twisted cubic). A_1, while not projectively normal, has degree 4, as does the volume of its Okounkov body. A_2 gives a graded linear seires of $\mathscr{O}_{\mathbb{P}^1}(3)$ where A_3 is $R(\mathscr{O}_{\mathbb{P}^1}(3))$ itself. In each case, they have degree 3.

2.3 Applications

Recall that a k-subalgebra of a section ring line bundle $\mathscr{L}$ is called a **graded linear series of** $\mathscr{L}$. For a (Cartier) divisor D on X, or more generally for W a graded linear series of D, the **Kodaira-Iitaka dimension** is q if $\dim_k W_t$ grows like t^q.

A (Cartier) divisor D on X is a **big divisor** if its Kodaira-Iitaka dimension is $n = \dim X$. The volume (or degree) of a big divisor is its self-intersection number $\operatorname{vol}(D) = \int_X (c_1(D))^n$. It is a consequence of Grothendieck-Hirzebruch-Riemann-Roch theorem that $\operatorname{vol}(D) = n! \lim_{t\to\infty} \frac{h^0(tD)}{t^n}$. Combining Theorem 1.1.5 with Theorem 2.1.4, we obatin

Corollary 2.3.1. [KKh12, Corollary 3.11(1)] (or [LM08, Theorem 2.3, 2.13]). Let $\mathscr{L}$ be a line bundle on X and $W \subset R(\mathscr{L})$ a graded linear series of Kodaira-Iitaka dimension q. Let $S = S(W)$ be its non-negative semigroup, $m = [\mathbb{Z} : \pi_M(G(S))]$, and $c = [\partial M, G(S) \cap \partial M]$. Lastly, let $\Delta(W)$ be the Newton-Okounkov body of $S(W)$. Then

$$\lim_{t\to\infty} \frac{\dim W_{mt}}{t^q} = \frac{1}{c}\operatorname{vol}(\Delta(W)).$$

[LM08, Lemma 2.3] implies that $G(S) = \mathbb{Z}^{n+1}$ for S a semigroup associated to (the section ring of) a big divisor. Hence, this such cases the subgroup indices $m = [\mathbb{Z}, \pi_M(G(S))]$ and $c = [\partial M, \partial M \cap G(S)]$ are both 1. Moreover, [LM08, Proposition 4.1] implies that the Newton-Okounkov body $\Delta(D)$ of a big divisor D only depends on the numerical equivalence class of D (once the admissible flag is fixed). Lastly, $\Delta(D)$'s over various D patch together nicely to give a universal Newton-Okounkov body in the following sense.

Theorem 2.3.2. [LM08, Theorem 4.5] Let $N^1(X)$ be numerical equivalence classes of divisors, and $p : \mathbb{R}^n \times N^1(X)_{\mathbb{R}} \to N^1(\mathbb{R})$ the projection map. Then there exists a convex body $\Delta(X) \subset \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$ such that for each class ξ of big divisor in $N^1(X)_{\mathbb{Q}}$ one has $p^{-1}(\xi) \cap \Delta(X) = \Delta(\xi)$.

This in particular implies that the map $\mathrm{vol} : \mathrm{Big}(X) \to \mathbb{R}$ is continuous ([LM08, Corollary 4.12]).

By analogy to the volume of a big divisor, we thus define the volume of a graded linear series to be $\mathrm{vol}(W) := \frac{q!}{c}\,\mathrm{vol}(\Delta(W))$. Using the classical Brunn-Minkowski inequality on the Newton-Okounkov bodies, we have the following.

Corollary 2.3.3. For graded linear series W_1, W_2 with the same q, c, and $m = 1$, we have

$$\mathrm{vol}(W_1)^{1/n} + \mathrm{vol}(W_2)^{1/n} \leq \mathrm{vol}(W_1W_2)^{1/n}.$$

In particular, for Cartier divisors D_1, D_2, we have

$$\mathrm{vol}(D_1)^{1/n} + \mathrm{vol}(D_2)^{1/n} \leq \mathrm{vol}(D_1 + D_2)^{1/n}.$$

Fujita approximation essentially states that volume of any big divisor on a projective can be approximated that of a ample divisor.

Theorem 2.3.4 (Fujita approximation)**.** [Laz04, Theorem 11.4.4] Suppose that D is a big integral divisor on projective X, and fix $\epsilon > 0$. Then there exists a birational map $\phi : X'?X$ and a decomposition $\phi^*(D) = A + E$ with A amples and E effective, such that $\mathrm{vol}_{X'}(A) > \mathrm{vol}_X(D) - \epsilon$.

Combining the machinery build in this section with Theorem 2.1.4, we have the following version of the Fujita approximation theorem:

Corollary 2.3.5. Let conditions be as in Corollary 2.3.1 with D a big divisor. Then

$$\frac{1}{c}\,\mathrm{vol}(D) = \frac{1}{c}\lim_{k\to\infty}\frac{\mathrm{vol}(mkD)}{k^q}$$

A classical theorem of Kushnirenko states the following. Consider for $A = \{a_1, \ldots, a_s\} \subset \mathbb{Z}^n$, the space of Laurent polynomials $\mathbb{C}^A := \{\sum_i c_i x^{a_i}\} \subset \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$. Suppose the affine $\mathbb{Z}$-span of A is $\mathbb{Z}^n$. Then for a general choice of n polynomials $\{f_1, \ldots, f_n\} \subset \mathbb{C}^A$, we have that the number of solutions in $(\mathbb{C}^*)^n$ of $f_1 = 0, \ldots, f_n = 0$ is the volume of $\mathrm{Conv}(A)$. The main idea behind the proof is the interpret elements of $\mathbb{C}^A$ as $H^0(X_A, \mathscr{L})$ where X_A is the toric variety of A and $\mathscr{L}$ is the very ample line bundle giving the embedding $X_A \subset \mathbb{P}^{s-1}$. One can similarly define suitably an intersection index $[L_1, \ldots, L_n]$ for any $L_i \subset K(X)$ finite dimensional. For details see [KKh12, 4.5-4.7]. Under this generalization, we have the generlized version of the Kushnirenko theorem:

Corollary 2.3.6. Let L be a graded linear series of a line bundle $\mathscr{L}$, and suppose that the Kodaira map $\Phi_L : X \dashrightarrow Y_L \subset \mathbb{P}(L)$ satisfies $\dim X = \dim Y_L$. Then we have

$$[L, \ldots, L] = n!\deg\Phi_L\frac{1}{c}\,\mathrm{vol}(\Delta(R_L))$$

(where c is again the index $[\partial M, \partial M \cap G(S(R_L))]$).

3 SAGBI bases and toric degenerations

Let A be a subalgebra of $\mathbb{C}[x_1,\ldots,x_n]$, and let ν be the restriction of Gröbner valuation on $\mathbb{C}[x_1,\ldots,x_n]$ to A. When the semigroup $S(A,\nu)$ is finitely generated, a collection $\{f_1,\ldots,f_s\}$ such that $\{\nu(f_1),\ldots,\nu(f_s)\}$ generates $S(A,\nu)$ is called **SAGBI basis** (Subalgebra analogue of Gröbner basis for Ideals) of A with respect to a valuation ν. Given a SAGBI basis $\{f_1,\ldots,f_s\}$, any $h \in A$ can be expressed as a polynomial in the f_i's by the *subduction algorithm*:

1. Write $\nu(h) = d_1\nu(f_1) + \cdots + d_s\nu(f_s)$ for some d_i's in $\mathbb{N}$.
2. Comparing the coefficient of the leading terms of $f_1^{d_1}\cdots f_r^{d_r}$ and h, we obtains $c \in \mathbb{C}$ such that $g = h - cf_1^{d_1}\cdots f_r^{d_r}$ cancels out the leading term and hence $\nu(g) < \nu(h)$.
3. Repeat the process with g until 0.

We can now generalize this notion to a large class of graded algebras.

Definition 3.0.7. Let A be a graded algebra with a graded valuation $\nu : A \to \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$ with one-dimensional leaves. Then $\{f_1,\ldots,f_s\} \subset A$ is a **SAGBI basis** for (A,ν) if $\{\nu(f_1),\ldots,\nu(f_s)\}$ generate the semigroup $S(A,\nu)$.

Note that A admits SAGBI basis iff $S(A,\nu)$ is finitely generated. It is not hard to check that the subduction algorithm still terminates in this more general case ([Kav15, Proposition 7.2]).

In the case when R is a subalgebra of the section ring $R(\mathscr{L})$ of a very ample line bundle $\mathscr{L}$ on a projective k-variety X, existence of a SAGBI basis gives a toric degeneration of X as follows. (Take $\widetilde{\nu}$ to be a valuation coming from a construction in Example 2.1.2).

Theorem 3.0.8. [And13, Proposition 5.1] Consider the filtration $F_\bullet$ on R given by $F_{\geq\alpha}$, and let $\operatorname{gr} R$ be the graded ring of R associated to the filtration $F_\bullet$. Suppose $S(R,\widetilde{\nu})$ is finitely generated. Then $\operatorname{gr} R \simeq k[S(R,\widetilde{\nu})]$, and moreover, there is a finitely generated, graded flat $k[t]$-subalgebra $\mathcal{R}$ of $R[t]$, such that $\mathcal{R}/t\mathcal{R} \simeq \operatorname{gr} R$ and $\mathcal{R}[t^{-1}] \simeq R[t^{\pm}]$.

Moreover, [And13, Proposition 5.16] shows that the construction of $\mathcal{R}$ is functorial. As an immediately corollary, taking R to be the section ring $R(\mathscr{L})$ itself of a very ample line bundle $\mathscr{L}$, we have the following toric degeneration statement.

Corollary 3.0.9. If X is a projective variety with a line bundle $\mathscr{L}$ such that $S = S(R(\mathscr{L}),\widetilde{\nu})$ is finitely generated, then there exists toric degeneration (a flat family of varieties) $\pi : \mathcal{X} \to \mathbb{A}^1$ such that $\pi^{-1}(0) \simeq \operatorname{Proj} k[S]$ and $\pi^{-1}(\lambda) \simeq X$ for $\lambda \neq 0$.

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