LOG-CONCAVE RAINBOWS AND WHERE TO FIND THEM

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ABSTRACT. We prove a log-concavity statement concerning the number of colored transversals. Along the way, we will encounter several important players in algebraic combinatorics, including mixed volumes, polymatroids, Chow rings, permutohedral/stellahedral fans, and more. These notes were prepared for Summer School in Algebraic Combinatorics at MPI Leipzig 2024.

1. INTRODUCTION

Let $[n] = \{1, \ldots, n\}$ for an integer $n \geq 1$. Let E be a finite set with a partition $E = E_1 \sqcup \cdots \sqcup E_n$, that is, let $\pi : E \twoheadrightarrow [n]$ be a surjective map and denote $E_i := \pi^{-1}(i)$.

Definition 1.1. A subset $S \subseteq E$ is *π-colored* if $|S \cap E_i| \leq 1$ for all $i \in [n]$. A *π-transversal* is a maximal π-colored subset, i.e. a subset $T \subseteq E$ such that $|T \cap E_i| = 1$ for all $i \in [n]$. Denote by 2^π the set of all π-colored subsets.

The primary combinatorial quantity we study is the following.

Definition 1.2. For a sequence $\boldsymbol{S} = (S_1, \ldots, S_m)$ of π-colored subsets of E, and for a sequence $\boldsymbol{d} = (d_1, \ldots, d_m) \in \mathbb{Z}^m_{\geq 0}$ such that $|\boldsymbol{d}| := d_1 + \cdots + d_m = n$, define the *$\pi$-capacity* of $(\boldsymbol{S}, \boldsymbol{d})$ to be

$$C(\boldsymbol{S}, \boldsymbol{d}) := |\{T \subseteq E \text{ a } \pi\text{-transversal} \mid \exists\, \varphi \colon T \to [m] \text{ with } \varphi^{-1}(j) \subseteq S_j \text{ and } |\varphi^{-1}(j)| = d_j \ \forall j \in [m]\}|.$$

Caution. Note that $C(\boldsymbol{S}, \boldsymbol{d})$ counts the number of π-transversals T admitting such a map φ, *not* the number of such maps. Given T, there can be several different φ satisfying the imposed condition.

Remark 1.3 (Flavor text)**.** Here is a "real life scenario" behind these definitions. Suppose we have an international meeting of n counties, each country sending delegates E_i. The delegates belong to various committees S_j where each country is represented at most once. Let's call a *panel* to be a subset of delegates consisting of exactly one delegates from each country (i.e. a π-transversal). Given a sequence $(d_1, \ldots, d_m)$ of nonnegative integers, one for each committee S_j, how many panels can be created by selecting d_j members from each committee S_j?

Example 1.4. Two examples with $n = 3$ and $m = 3$ are depicted below.

For $\boldsymbol{d} \in \mathbb{Z}^m_{\geq 0}$, write $\dfrac{\boldsymbol{x}^{\boldsymbol{d}}}{\boldsymbol{d}!} := \dfrac{x_1^{d_1} \cdots x_m^{d_m}}{d_1! \cdots d_m!}$. The following is our main theorem about π-capacities.

Theorem A. For π and $\boldsymbol{S}$, the *π-capacity polynomial* $f \in \mathbb{R}[x_1, \ldots, x_m]$ defined by

$$f(\boldsymbol{x}) = \sum_{\substack{\boldsymbol{d} \in \mathbb{Z}^m_{\geq 0} \\ |\boldsymbol{d}| = n}} C(\boldsymbol{S}, \boldsymbol{d}) \frac{\boldsymbol{x}^{\boldsymbol{d}}}{\boldsymbol{d}!}$$

is *Lorentzian* in the sense of [BH20].

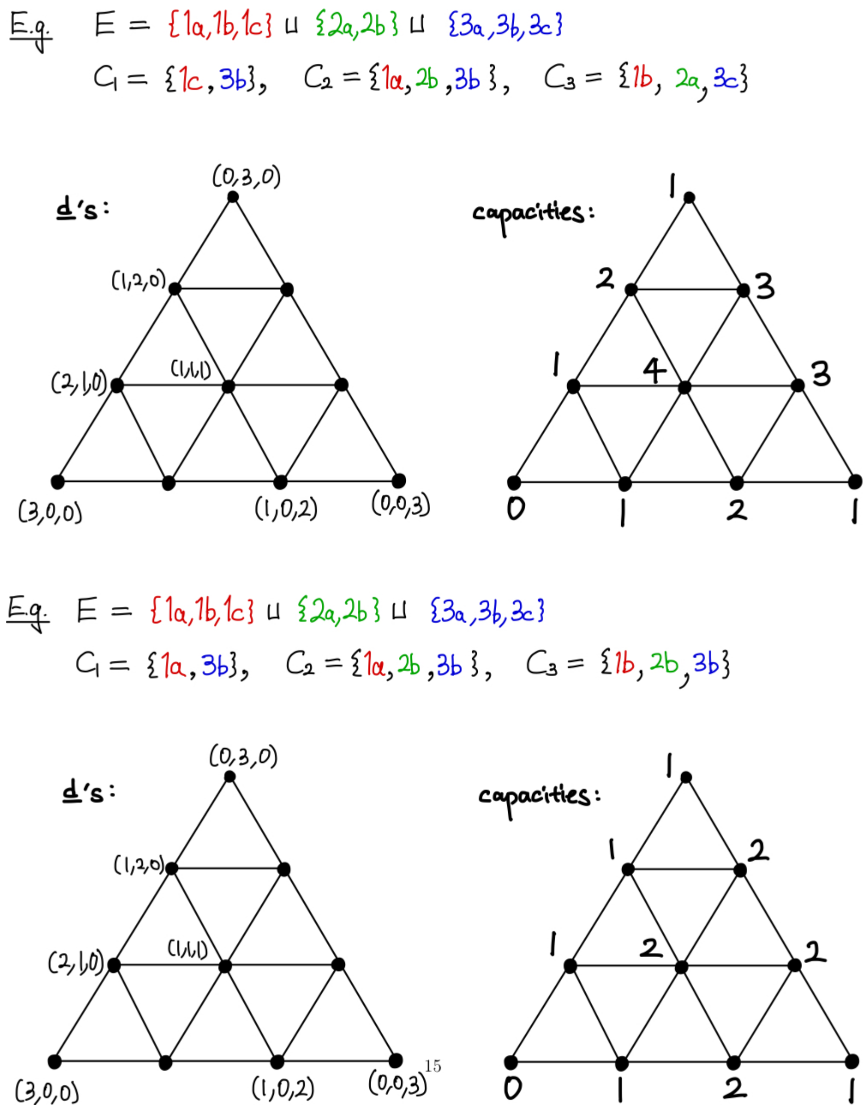

FIGURE 1. Note that in the second example, for $\boldsymbol{d} = (1, 2, 0)$, the only transversal possible is $\{1a, 2b, 3b\}$, although it can be created in two different ways.

We defer the definition of Lorentzian polynomials to Section 3, and only mention here a log-concavity behavior that Lorentzian polynomials enjoy. A nonnegative sequence $(a_0, a_1, \dots, a_\ell)$ is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq \ell - 1$, and it has *no internal zeros* if $a_i \neq 0$ and $a_j \neq 0$ implies $a_k \neq 0$ for all $i < k < j$.

Proposition 1.5. [BH20, Example 2.26] The coefficients $c_{\boldsymbol{d}}$ of a Lorentzian polynomial $g(\boldsymbol{x}) = \sum_{\boldsymbol{d}} c_{\boldsymbol{d}} \frac{\boldsymbol{x}^{\boldsymbol{d}}}{\boldsymbol{d}!}$, when read along any $\mathbf{e}_i - \mathbf{e}_j$ direction, form a log-concave sequence with no internal zeros.

The ideas and the methods behind the proof of the main theorem comprise of interactions between various topics of interest in algebraic combinatorics—polytopes, poly- and multi- matroids, tropical Hodge theory, toric varieties, and moduli spaces of pointed rational curves.

Remark 1.6. A sketch of the proof of the main theorem is as follows. Consider $(\mathbb{P}^1)^n$, where the i-th copy of $\mathbb{P}^1$ has $|E_i|$ many distinct markings. These markings make a "grid pattern" on $(\mathbb{P}^1)^n$. Let X be the sequential blow-up of all the strata in the grid pattern starting with the lowest dimensional ones. Alternatively, for each π-colored subset S, one has a rational map $(\mathbb{P}^1)^n \dashrightarrow \mathbb{P}(\mathbb{C}^S \oplus \mathbb{C})$, and X resolves the indeterminacy. Let h_S be the pullback to X of the hyperplane class in $\mathbb{P}(\mathbb{C}^S \oplus \mathbb{C})$. Then, one has that

$$\int_X h_{S_1} \cdots h_{S_n} = C((S_1, \ldots, S_n), (1, \ldots, 1)).$$

This is the hardest step (with no known "algebro-geometric proof"), but given this, the rest follows from the general theory of Lorentzian polynomials — namely, that intersection numbers of base-point-free divisors give rise to Lorentzian polynomials [BH20, Theorem 4.6].

2. Simplices, matchings, and polymatroids

Let $\mathbf{e}_i$ denote the i-th standard basis vector of $\mathbb{R}^n$. For a subset $S \subseteq [n]$, denote $\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i$, and for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, denote $x_S := \sum_{i\in S} x_i$. We begin with some classical facts about volumes of polytopes in $\mathbb{R}^n$. Recall that the *Minkowski sum* of two polytopes P and Q in $\mathbb{R}^n$ is the pointwise sum, i.e.

$$P + Q = \{x + y \in \mathbb{R}^n : x \in P \text{ and } y \in Q\}.$$

Let $\mathrm{Volume}(P)$ denote the volume of P, normalized so that the unit cube $[0, 1]^n$ gets volume $n!$. (i.e. The standard n-dimensional simplex in $\mathbb{R}^n$ gets volume 1).

Fact 2.1 (Minkowski)**.** For polytopes $P_1, \ldots, P_m$ in $\mathbb{R}^n$, the function $\mathrm{Volume}(x_1P_1 + \cdots + x_mP_m)$ is a degree n homogeneous polynomial in $x_1, \ldots, x_m$.

Definition 2.2. For polytopes $P_1, \ldots, P_n$ in $\mathbb{R}^n$, define their *mixed volume* to be

$$MV(P_1, \ldots, P_n) := \frac{1}{n!} \cdot (\text{the coefficient of } x_1 \cdots x_n \text{ in } \mathrm{Volume}(x_1P_1 + \cdots + x_nP_n)).$$

Exercise 2.3 (Sanity check)**.** Let $P = \mathrm{conv}(0, \mathbf{e}_1, \mathbf{e}_2)$ and $Q = \mathrm{conv}(0, -\mathbf{e}_1, \mathbf{e}_2)$ in $\mathbb{R}^2$. Compute $\mathrm{Volume}(xP + yQ)$, and verify that $MV(P, Q) = 2$.

Exercise 2.4. Write P^d for the sequence of P repeated d times. For $\boldsymbol{d} \in \mathbb{Z}^m_{\geq 0}$ such that $|\boldsymbol{d}| = n$, verify that $MV({P_1}^{d_1}, \ldots, {P_m}^{d_m}) =$ the coefficient of $\binom{n}{d_1,\ldots,d_m} x_1^{d_1} \cdots x_m^{d_m}$ in the polynomial $\mathrm{Volume}(x_1P_1+ \cdots + x_mP_m)$. In particular, conclude that $MV(P^n) = \mathrm{Volume}(P)$.

We now consider mixed volumes of standard simplices. For a subset $S \subseteq [n]$, denote by

$$\Delta^0_S = \mathrm{conv}(\{0\} \cup \{\mathbf{e}_i : i \in S\}).$$

Theorem 2.5. For a sequence $(S_1, \ldots, S_n)$ of subsets of $[n]$, we have

$$MV(\Delta^0_{S_1}, \ldots, \Delta^0_{S_n}) = \begin{cases} 1 & \text{if } |\bigcup_{i\in I} S_i| \geq |I| \text{ for all } I \subseteq [n] \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof here is a modification of the proof of a related result [Pos09, Theorem 9.3]. Let us recall the Bernstein–Kushnirenko–Khovanskii (BKK) theorem, which states the following. Given lattice polytopes $P_1, \dots, P_n$ in $\mathbb{R}^n$, for each $i \in [n]$, let $f_i = \sum_{\boldsymbol{a} \in P_i \cap \mathbb{Z}^n} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}}$ be a Laurent polynomial with a generic choice of the coefficients $c_{\boldsymbol{a}}$. Then, one has

$$MV(P_1, \dots, P_n) = |\{\boldsymbol{t} \in (\mathbb{C}^*)^n : f_1(\boldsymbol{t}) = \cdots = f_n(\boldsymbol{t}) = 0\}|.$$

Here $\mathbb{C}^*$ denotes $\mathbb{C} \setminus \{0\}$. Applying the BKK theorem to the case of standard simplices, one concludes the desired statement by recalling the Hall's marriage theorem. □

Exercise 2.6. Fill in the details for the last sentence of the proof of Theorem 2.5.

Exercise 2.7 (For those familiar with toric geometry)**.** Prove the BKK theorem.

Studying Minkowski sums of standard simplices leads to the following central notion in discrete convex optimization.

Definition 2.8. A *polymatroid* on $[n]$ is the data of a function $\mathrm{rk} : 2^{[n]} \to \mathbb{R}$, called its *rank function*, that satisfies

(1) $\mathrm{rk}(\emptyset) = 0$,
(2) (Monotone) $\mathrm{rk}(S_1) \leq \mathrm{rk}(S_2)$ if $S_1 \subseteq S_2 \subseteq [n]$, and
(3) (Submodular) $\mathrm{rk}(S_1) + \mathrm{rk}(S_2) \geq \mathrm{rk}(S_1 \cup S_2) + \mathrm{rk}(S_1 \cap S_2)$ for all $S_1, S_2 \subseteq [n]$.

Its *rank* is $r := \mathrm{rk}([n])$, and its *independence polytope* $IP(\mathrm{rk})$ and its *base polytope* $BP(\mathrm{rk})$ are

$$IP(\mathrm{rk}) := \{x \in \mathbb{R}^n_{\geq 0} : x_S \leq \mathrm{rk}(S) \text{ for all } S \subseteq [n]\} \quad \text{and} \quad BP(\mathrm{rk}) := IP(\mathrm{rk}) \cap \{x_{[n]} = r\}.$$

An *integral polymatroid* is a polymatroid whose rank function rk takes values in $\mathbb{Z}$.

See [Edm70] for a treatment of polymatroids from the viewpoint of polytopes and matroids, and [Mur03] for a detailed treatment of discrete convex optimization.

Exercise 2.9. Classify all combinatorial types of independence polytopes of polymatroids on $[2]$.

Exercise 2.10. Show that a (nonnegative) Minkowski sum of standard simplices is the independence polytope of a polymatroid. Show that not every independence polytope of a polymatroid arise in this way. (You may postpone the exercise until Fact 3.8.(2) and Theorem 4.5 are available).

Remark 2.11. Base polytopes of polymatroids are also known as *generalized permutohedra* (that are contained in the nonnegative orthant). In [Pos09], Postnikov noted that every generalized permutohedra is a *signed* Minkowski sum of the simplices $\{\Delta_S\}_{S \subseteq [n]}$ where $\Delta_S = \mathrm{conv}\{\mathbf{e}_i : i \in S\}$.

Fact 2.12. For an integral polymatroid rk on $[n]$, the polytope $IP(\mathrm{rk})$ is a lattice polytope (and hence $BP(\mathrm{rk})$ is also) [Edm70, (8)]. Moreover, in this case $BP(\mathrm{rk}) \cap \mathbb{Z}^n$ is a *M-convex* set: a subset J of $\mathbb{Z}^n$ is said to be M-convex if for every $\alpha, \beta \in J$ with $\alpha_i > \beta_i$ for some $i \in [n]$, there exists $i \neq j \in [n]$ such that both $\alpha - \mathbf{e}_i + \mathbf{e}_j$ and $\beta + \mathbf{e}_j - \mathbf{e}_i$ are in J [Mur03, Theorem 4.15].

Exercise 2.13. Let the notations be as in the introduction, and recall the π-capacity polynomial $f(\boldsymbol{x}) = \sum_{\boldsymbol{d}} C(\boldsymbol{S}, \boldsymbol{d}) \frac{\boldsymbol{x}^{\boldsymbol{d}}}{\boldsymbol{d}!}$. Let $\mathrm{supp}(f) := \{\boldsymbol{d} \in \mathbb{Z}^m_{\geq 0} : c_{\boldsymbol{d}} \neq 0\}$ be its support. Show that for $\boldsymbol{a} \in \mathbb{Z}^m_{\geq 0}$ is in $\mathrm{supp}(f)$ if and only if $|\boldsymbol{a}| = n$ and $\boldsymbol{a}_I \leq |\pi(\bigcup_{i \in I} S_i)|$ for all $I \subseteq [m]$. Deduce that the support of the π-capacity polynomial f is M-convex.

3. Chow rings of fans

Let N and M be dual lattices of rank n, with the pairing $N \times M \to \mathbb{Z}$ denoted by $\langle \cdot, \cdot \rangle$. Let $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. For instance, throughout this section one may take $N = \mathbb{Z}^n$ and $M = (\mathbb{Z}^n)^\vee \simeq \mathbb{Z}^n$ (so $N_{\mathbb{R}} = \mathbb{R}^n = M_{\mathbb{R}}$), with the pairing $\langle \cdot, \cdot \rangle$ being the standard inner product.

Let Σ be a rational *fan* in $N_{\mathbb{R}}$, i.e. a collection of rational polyhedral cones closed under taking faces such that any two cones intersect in a face of each. Let $\Sigma(1)$ denote the set of its rays, and for $\rho \in \Sigma(1)$, denote by $u_\rho \in N$ its *primitive ray vector* (i.e. the first integral vector in the ray ρ). Unless otherwise specified, we will always assume a fan Σ to be *pure d-dimensional* (i.e. all maximal cones have dimension d), *lineality-less* (i.e. the minimal cone in the fan is the origin, not a linear space of positive dimension), and *smooth* (i.e. for every cone σ in Σ, its primitive ray vectors form a subset of a $\mathbb{Z}$-basis of N). In particular, Σ is *simplicial* (i.e. every cone σ has exactly $\dim(\sigma)$ many rays).

Definition 3.1. The *Chow ring* of Σ is the graded $\mathbb{R}$-algebra[1]

$$A^\bullet(\Sigma) = \frac{\mathbb{R}[x_\rho : \rho \in \Sigma(1)]}{I_\Sigma + J_\Sigma}$$

where I_Σ and J_Σ are ideals defined by

$$I_\Sigma = \left\langle \prod_{\rho \in S} x_\rho : S \subseteq \Sigma(1) \text{ not forming a cone in } \Sigma \right\rangle \quad \text{and} \quad J_\Sigma = \left\langle \sum_{\rho \in \Sigma(1)} \langle u_\rho, v \rangle x_\rho : v \in M \right\rangle.$$

A *divisor* on Σ is a linear combination $D = \sum_\rho c_\rho x_\rho$, whose *divisor class* $[D]$ is its image in $A^1(\Sigma)$.

Example 3.2 (skeletons of normal fans of opposite simplices)**.** Let $N = \mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E$ for E a finite set with $|E| = n+1$. Under the standard inner product on $\mathbb{Z}^E$, the dual lattice M is identified with $\mathbf{e}_E^\perp := \{v \in \mathbb{Z}^E : \langle \mathbf{e}_E, v \rangle = 0\} = \operatorname{span}(\mathbf{e}_i - \mathbf{e}_j : i \neq j \in E)$. For $0 \leq r \leq n$, let $\Sigma_{r,E}$ be the pure r-dimensional fan in $N_{\mathbb{R}}$ whose maximal cones are

$$\sigma_S = \operatorname{cone}\{\overline{\mathbf{e}}_i : i \in S\}$$

for $S \subsetneq E$ a subset of cardinality r. Then, we find that

$$A^\bullet(\Sigma_{r,E}) = \frac{\mathbb{R}[x_i : i \in E]}{\langle \prod_{i \in S'} x_i : |S'| = r+1 \rangle + \langle x_i - x_j : i \neq j \in E \rangle} \simeq \mathbb{R}[h]/\langle h^{r+1} \rangle.$$

Observe that $\Sigma_{n,E}$ is the (outer) normal fan of the opposite simplex $\operatorname{conv}\{-\mathbf{e}_i : i \in E\}$ in $\mathbb{R}^E$, and $\Sigma_{r,E}$ is the r-skeleton of $\Sigma_{n,E}$.

Exercise 3.3. Show that $A^\bullet(\Sigma)$ is spanned by square-free monomials in the x_ρ. In particular, conclude that $A^i(\Sigma) = 0$ for $i > d$.

Definition 3.4. For a cone $\sigma \in \Sigma$, write $x_\sigma := \prod_{\rho \preceq \sigma} x_\rho$. We say that Σ is *balanced* if the assignment $x_\sigma \mapsto 1$ for every maximal cone σ of Σ defines a linear map $A^d(\Sigma) \to \mathbb{R}$, denoted $\deg_\Sigma$.

A fan Σ is *complete* if its support $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ equals $\mathbb{R}^n$.

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[1] We work with $\mathbb{R}$ coefficients here, although the original definition is over $\mathbb{Z}$. Over $\mathbb{Z}$, the Chow ring can have torsion.

Exercise 3.5. Show that a complete fan Σ is balanced. In fact, the map $\deg_\Sigma : A^n(\Sigma) \to \mathbb{R}$ is an isomorphism in this case. You may assume as a fact that $A^n(\Sigma) \neq 0$ when Σ is complete. Give an example of a fan that is not balanced.

Remark 3.6. For a complete fan Σ, the Chow ring of Σ is the cohomology ring of the toric variety X_Σ, and the degree map $\deg_\Sigma$ coincides with the Poincaré duality map $H^{2n}(X_\Sigma, \mathbb{R}) \xrightarrow{\sim} \mathbb{R}$. See [CLS11, Chapter 12].

We now assume Σ to be complete for the rest of this section. In classical algebraic geometry, intersection numbers of ample/nef divisors display a positivity behavior. In our context, the notion of nef divisors has the following polyhedral description.

Definition 3.7. A polytope $P \subset M_\mathbb{R}$ is a *deformation* of Σ if its outer normal fan Σ_P (which may have lineality and not be smooth) coarsens Σ. A divisor $D = \sum_\rho c_\rho x_\rho$ on Σ is *nef* if there is a deformation P of Σ such that $c_\rho = \max_{v \in P} \langle u_\rho, v \rangle$ for all ρ.

Fact 3.8. We collect some facts about nef divisors on (smooth, complete, lineality-less) fans. See [Ful93, Chapters 3 and 5] or [CLS11, Chapters 6 and 13].

(1) For a nef divisor $D = \sum_\rho c_\rho x_\rho$, such a deformation P is unique, namely,
$$P = \{v \in M_\mathbb{R} : \langle v, u_\rho \rangle \leq c_\rho \ \forall \rho \in \Sigma(1)\}.$$
We may thus write $D = D_P$.

(2) Sum of nef divisors corresponds to Minkowski sum of deformations, i.e. we have
$$D_{P_1} + D_{P_2} = D_{P_1 + P_2}.$$
In particular $[D_{P_1}] = [D_{P_2}]$ if and only if P_1 and P_2 are translates of each other.

(3) We have $\deg_\Sigma(D_P^n) = \mathrm{Volume}(P)$. In particular, for deformations $P_1, \ldots, P_m$ of Σ, the volume polynomial $\mathrm{Volume}(x_1 D_{P_1} + \cdots + x_m D_{P_m})$ of their Minkowski sums is a homogeneous polynomial of degree n in $x_1, \ldots, x_m$ (cf. Fact 2.1).

Exercise 3.9 (Sanity check)**.** (cf. Exercise 2.4) Verify that for $\boldsymbol{d} \in \mathbb{Z}^m$ satisfying $|\boldsymbol{d}| = n$, we have
$$MV({P_1}^{d_1}, \ldots, {P_m}^{d_m}) = \deg_\Sigma(D_{P_1}^{d_1} \cdots D_{P_m}^{d_m}).$$

Volume polynomials of polytopes are among the prototypical examples of Lorentzian polynomials. Let us recall the definition of Lorentzian polynomials introduced in [BH20], and independently in [ALOGV24] as *completely log-concave polynomials*.

Definition 3.10. A real homogeneous polynomial $g(\boldsymbol{x}) = \sum_{\boldsymbol{d}} c_{\boldsymbol{d}} \frac{\boldsymbol{x}^{\boldsymbol{d}}}{\boldsymbol{d}!} \in \mathbb{R}[x_1, \ldots, x_m]$ of degree n with nonnegative coefficients is said to be *Lorentzian* if:

(i) its support $\mathrm{supp}(g) := \{\boldsymbol{d} : c_{\boldsymbol{d}} \neq 0\}$ is M-convex, and

(ii) every $(n-2)$-th partial derivative $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{n-2}}} g$ of g is a quadratic form with at most one positive eigenvalue.

For an introduction to Lorentzian polynomials from the viewpoint of algebraic geometry, see Eur_IntroLorentzianPolynomials (link).

Exercise 3.11 (Sanity check)**.** Pick any one of the two the π-capacity polynomials in Example 1.4 and verify that it is indeed Lorentzian. (You may use a computer to compute eigenvalues).

Theorem 3.12. [BH20, Theorem 4.6] For nef divisors $D_1, \dots, D_m$ on Σ, the polynomial

$$\deg_\Sigma\left((x_1 D_1 + \cdots + x_m D_m)^n\right) = n! \cdot \sum_{\boldsymbol{d}} MV({P_1}^{d_1}, \dots, {P_m}^{d_m}) \frac{\boldsymbol{x}^{\boldsymbol{d}}}{\boldsymbol{d}!}$$

is Lorentzian. (Note that the equality of the two polynomials above is Exercise 3.9).

When $m = 2$ above, the above statement is the classical Aleksandrov–Fenchel inequalities for mixed volumes of convex bodies. The theorem is stated in [BH20] in a more general setting of nef divisors on projective varieties.

4. Stellahedral fans

We now return to polymatroids from the viewpoint of complete fans and nef divisors.

Definition 4.1. Define the **affine permutohedral fan** $\mathring{\Sigma}_n$ in $\mathbb{R}^n$ to be the fan consisting of cones

$$\sigma_{\mathscr{F}} := \text{cone}\{\mathbf{e}_{F_1}, \dots, \mathbf{e}_{F_k}\}$$

for every chain $\mathscr{F} : \emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subseteq [n]$ of nonempty (not necessarily proper) subsets of $[n]$.

Let us observe some features of the affine permutohedral fan:

(1) The primitive rays of $\mathring{\Sigma}_n$ are the $\{\mathbf{e}_S : \emptyset \subsetneq S \subseteq [n]\}$.
(2) For every subset $S \subseteq [n]$, the restriction of $\mathring{\Sigma}_n$ to the coordinate subspace $\mathbb{R}^S \subseteq \mathbb{R}^n$ is another copy of the affine permutohedral fan $\mathring{\Sigma}_{|S|}$.
(3) This fan is the full barycentric subdivision of the nonnegative orthant. More precisely, this fan is obtained by performing stellar subdivisions of all the cones (in the order of decreasing dimension) in the fan whose maximal cone is $\mathbb{R}^n_{\geq 0}$.

This fan is where "all the action is going to happen," but is not complete. It admits the following distinguished completion.

Definition 4.2. Define the **stellahedral fan** Σ_n in $\mathbb{R}^n$ to be the fan consisting of cones

$$\sigma_{(\mathscr{F}, I)} := \text{cone}\left(\{-\mathbf{e}_i : i \in I\} \cup \{\mathbf{e}_F : F \in \mathscr{F}\}\right)$$

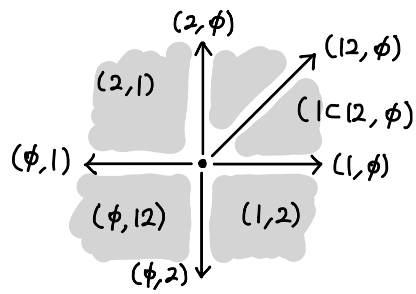
for every pair $(\mathscr{F}, I)$ of a chain $\mathscr{F} : \emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subseteq [n]$ of nonempty subsets of $[n]$ and a subset $\emptyset \subseteq I \subseteq [n]$ such that $I \cap F = \emptyset$ for all $F \in \mathscr{F}$.

Example 4.3. Σ_2 is illustrated below.

Let us observe some features of the stellahedral fan:

(1) Considering the pairs $(\mathscr{F}, \emptyset)$, we find that the stellahedral fan contains the affine permutohedral fan as its "$\mathbb{R}^n_{\geq 0}$ part."
(2) The primitive rays of Σ_n are the primitive rays of $\mathring{\Sigma}_n$ along with $\{-\mathbf{e}_i : i \in [n]\}$.

For a subset $\emptyset \subsetneq S \subseteq [n]$, let us denote x_S to be the variable in the Chow ring $A^\bullet(\Sigma_n)$ corresponding to the ray of $\mathbf{e}_S$.



Exercise 4.4 (Sanity check)**.** Verify that $\{x_S : \emptyset \subsetneq S \subseteq [n]\}$ is a basis of $A^1(\Sigma_n)$.

Theorem 4.5. [EHL23, Proposition 3.13] The assignment $\mathrm{rk} \mapsto \sum_S \mathrm{rk}(S) x_S$ gives a bijection between polymatroids on $[n]$ and nef divisor classes on Σ_n.

Exercise 4.6. Show that the nef divisors from standard simplices form a basis for $A^1(\Sigma_n)$ as follows.

(1) Show that for $S \subseteq [n]$, the function

$$\mathrm{rk}(A) = \begin{cases} 1 & \text{if } A \cap S \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is a polymatroid whose independence polytope is the simplex Δ^0_S. In other words, by Theorem 4.5, the nef divisor on Σ_n corresponding to Δ^0_S is $\sum_{A \cap S \neq 0} x_A$.

(2) For $\emptyset \subsetneq S \subseteq [n]$, define $h_S := \sum_{A \cap S \neq 0} x_A$. Show that this defines an invertible linear change of variables, so that $\{h_S : \emptyset \subsetneq S \subseteq [n]\}$ is a basis of $A^1(\Sigma_n)$ (since the x_S's form a basis). It may help to go through an intermediate basis $\{y_S : \emptyset \subsetneq S \subseteq [n]\}$, where

$$y_{[n]} := \sum_{A \subseteq [n]} x_A \quad \text{and} \quad y_S := -x_{[n] \setminus S} \text{ for nonempty } S \subsetneq [n].$$

Proof of Theorem A when $|E_i| = 1$. For a subset $S \subseteq E$, let D_S be the nef divisor on Σ_n corresponding to the simplex Δ^0_S, which is a deformation of Σ_n. Then, by Theorem 2.5, we find that the capacity polynomial times $n!$ is the volume polynomial

$$\deg_{\Sigma_n}\left((x_1 D_{S_1} + \cdots + x_m D_{S_m})^n\right),$$

which is Lorentzian by Theorem 3.12. □

5. Multi-permutohedral fans

Let us now return to the π-colored setting. Recall the setup: we had $\pi : E \twoheadrightarrow [n]$ and $E_i := \pi^{-1}(i)$. Let us denote

$$N^\pi := \mathbb{Z}^{E_1}/\mathbb{Z}\mathbf{e}_{E_1} \times \cdots \times \mathbb{Z}^{E_n}/\mathbb{Z}\mathbf{e}_{E_n},$$

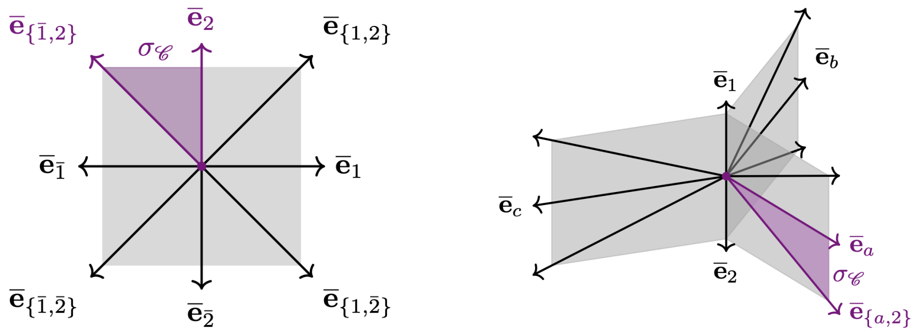
and for a subset $F \subseteq E$, denote by $\overline{\mathbf{e}}_F$ the image of $\mathbf{e}_F \in \mathbb{Z}^E$ in the quotient N^π.

Definition 5.1. Define the π-**multipermutohedral fan** Σ_π to be the fan in $N^\pi_{\mathbb{R}}$ consisting of cones

$$\sigma_{\mathscr{F}} = \mathrm{cone}\{\overline{\mathbf{e}}_F : F \in \mathscr{F}\}$$

for every chain $\mathscr{F}$ of nonempty π-colored subsets.

Example 5.2. We feature two examples with $n = 2$. In the first, we have $E_1 = \{1, \bar{1}\}$ and $E_2 = \{2, \bar{2}\}$. In the second, we have $E_1 = \{a, b, c\}$ and $E_2 = \{1, 2\}$.



Let us observe some features of the multipermutohedral fan.

(1) The fan Σ_π is pure n-dimensional, lineality-less, and smooth respect to the lattice N^π.

(2) For each π-colored subset S, consider the subspace $\mathbb{R}^S$ of $\mathbb{R}^\pi$ defined by
$$\mathbb{R}^S := \operatorname{span}\{\overline{\mathbf{e}}_i : i \in S\} \simeq \mathbb{R}^{|S|}.$$
Every cone of Σ_π meets every $\mathbb{R}^S$ in a face, and moreover, under the obvious $\mathbb{R}^S \simeq \mathbb{R}^{|S|}$, we have that Σ_π restricted to $\mathbb{R}^S_{\geq 0}$ is a copy of the affine permutohedral fan $\mathring{\Sigma}_{|S|}$.

(2′) In particular, the support of Σ_n is equal to the support of the product fan $\Sigma_{1,E_1} \times \cdots \times \Sigma_{1,E_n}$ (where Σ_{1,E_i} is the fan given in Example 3.2).

Exercise 5.3. Show that the Chow ring of Σ_π has the following presentation:
$$A^\bullet(\Sigma_\pi) = \frac{\mathbb{R}[x_S : S \text{ a nonempty } \pi\text{-colored subset}]}{\langle x_S x_{S'} : S \not\subseteq S' \text{ and } S \not\supseteq S'\rangle + \langle \sum_{S \ni i} x_S - \sum_{S' \ni j} x_{S'} : i \neq j \in E_k \text{ for some } k \in [n]\rangle}.$$

Remark 5.4. Is the ring $A^\bullet(\Sigma_\pi)$ Koszul? (I don't know).

The relation between standard simplices and nef generators of $A^\bullet(\Sigma_n)$ in the case of stellahedral fans generalizes to multipermutohedral fans in the following way.

Definition 5.5. For a π-colored subset S, define $h_S \in A^1(\Sigma_\pi)$ by
$$h_S = \sum_{S' \cap S \neq \emptyset} x_{S'}.$$

Exercise 5.6. Deduce Theorem A when $|E_i| = 2$ for all $i \in [n]$. The following observations may be useful. Assume throughout that $|E_i| = 2$ for all $i \in [n]$.

(i) In this case, by identifying $\mathbb{R}^{E_i}/\mathbb{R}\mathbf{e}_{E_i}$ with $\mathbb{R}$ for each i, note that Σ_π is a complete fan in $\mathbb{R}^n$, whose 2^n orthants correspond to the π-transversals.

(ii) Each h_S is a nef divisor; what is the corresponding polytope in $\mathbb{R}^n$?

(iii) Note that every π-transversal corresponds to an orthant in $\mathbb{R}^n$, and apply Theorem 2.5 "in each orthant."

So far, we could prove Theorem A when $|E_i| \leq 2$ for all $i \in [n]$ by "classical" methods involving nef divisors on complete fans, their deformations, mixed volumes, and polymatroids. To obtain Theorem A in full generality, we will need two relatively new tools: tropical Hodge theory and normal complexes (generalizing to incomplete fans the positivity properties and deformations of nef divisors on complete fans).

6. A Glimpse of tropical Hodge theory

Exercise 6.1 (Optional)**.** Show that a divisor D on a complete fan Σ is nef if and only if for every cone $\sigma \in \Sigma$, there exists a divisor $D' = \sum_\rho c_\rho x_\rho$ representing the same divisor class $[D]$ satisfying the following: (i) $c_\rho = 0$ for all rays ρ of σ, and (ii) $c_\rho \geq 0$ for all rays ρ not in σ such that $\sigma \cup \rho$ forms a cone in Σ.

We no longer assume Σ to be complete now. For incomplete fans, we take the characterization of nefness on complete fans in Exercise 6.1 as the definition of nefness.

Definition 6.2. A divisor D on Σ is said to be *nef* if for every cone $\sigma \in \Sigma$, there exists a divisor $D' = \sum_\rho c_\rho x_\rho$ representing the same divisor class $[D]$ satisfying the following: (i) $c_\rho = 0$ for all rays ρ of σ, and (ii) $c_\rho \geq 0$ for all rays ρ not in σ such that $\sigma \cup \rho$ forms a cone in Σ. When D' can be made to have $c_\rho > 0$ in condition (ii) for all σ, we say that D is *ample*.

A pure-dimensional, smooth, balanced fan is *Lefschetz* if it and all its star fans satisfy (the mixed version of) the Kähler package consisting of Poincare duality, hard Lefschetz, and Hodge-Riemann relations. We will not need the precise definition of Lefschetz fans, which can be found in [ADH23, Section 5].

Fact 6.3. We collect some facts about Lefschetz fans. For proofs see [ADH23, Section 5].

(1) The same statement as in Theorem 3.12 holds for Lefschetz fans.
(2) A product of Lefschetz fans is Lefschetz.
(3) A stellar subdivision of a Lefschetz fan is Lefschetz.

These facts, together with the easy verification that the fan $\Sigma_{1,E}$ (Example 3.2) is Lefschetz, implies that the multi-permutohedral fan is Lefschetz. Moreover, we have the following.

Exercise 6.4. Show that the divisors h_S on the multipermutohedral fan Σ_π are nef.

Hence, we deduce the following.

Proposition 6.5. The polynomial

$$\deg_{\Sigma_\pi}\left((x_1 h_{S_1} + \cdots + x_m h_{S_m})^n\right) \in \mathbb{R}[x_1, \ldots, x_m]$$

is Lorentzian for any π-colored subsets $S_1, \ldots, S_m$ of E.

Due to this proposition, for the proof of Theorem A, we are done once we show that $n!$ times the π-capacity polynomial equals the polynomial $\deg_{\Sigma_\pi}\left((x_1 h_{S_1} + \cdots + x_m h_{S_m})^n\right)$. In the next section, we establish this following [CDE+24].

7. Multi-polymatroids and their independence complexes

Let us first describe a generalization of Fact 3.8.(3) to the setting of not necessarily complete fans, given by Nathanson and Ross [NR23]. Unlike the complete case, it requires a choice of an inner product $*$ on $N_{\mathbb{R}}$. Let Σ be a (pure d-dimensional, lineality-less, and smooth) fan in $N_{\mathbb{R}}$.

Definition 7.1. A divisor $D = \sum_\rho c_\rho x_\rho$ on Σ is said to be $*$-*pseudo-cubical* if for every cone $\sigma \in \Sigma$,

$$\sigma \cap \{v \in N_{\mathbb{R}} : v * u_\rho = c_\rho \text{ for all rays } \rho \text{ of } \sigma\} \quad \text{is nonempty,}$$

and D is further $*$-*cubical* if the nonempty intersection above in the relative $\operatorname{relint}(\sigma)$ of σ. For a $*$-pseudo-cubical divisor D on Σ, its *normal complex* $P_{*,\Sigma}(D)$ is the union

$$P_{*,\Sigma}(D) := \bigcup_{\sigma\in\Sigma} P_{*,\sigma}(D) \quad \text{where} \quad P_{*,\sigma}(D) := \sigma \cap \{N_{\mathbb{R}} : v * u_\rho \le c_\rho \text{ for all rays } \rho \text{ of } \sigma\}.$$

We will drop the "$*$-" when the inner product is clear in context.

Example 7.2. With the standard inner product on $\mathbb{R}^2$, the divisor $D_1 = 2x_1 + 2x_2 + 3x_{12}$ on the stellahedral fan Σ_2 is a cubical. The divisor $D_2 = x_1 + x_2 + x_{12}$ is pseudo-cubical. The divisor $D_3 = 2x_1 + x_2 + 3x_{12}$ is nef but not pseudo-cubical.

Exercise 7.3. With the standard inner product on $\mathbb{R}^n$, show that a divisor $D = \sum_{\emptyset \subsetneq S \subseteq [n]} c_S x_S$ on the affine permutohedral fan $\mathring{\Sigma}_n$ is pseudo-cubical if and only if for all $\emptyset \subseteq S \subsetneq S' \subsetneq S'' \subseteq [n]$ such that $|S'' \setminus S'| = |S' \setminus S| = 1$,

$$c_S \le c_{S'} \quad \text{and} \quad c_{S'} - c_S \ge c_{S''} - c_{S'},$$

with the convention that $c_\emptyset = 0$. (The divisor is further cubical if both inequalities above is strict).

For each maximal cone $\sigma \in \Sigma_{\max}$, the volume $\operatorname{Vol}_{*,\sigma}$ on $\operatorname{span}_{\mathbb{R}}(\sigma)$ is normalized as follows. Let $N(\sigma)$ be the sublattice $N \cap \operatorname{span}_{\mathbb{R}}(\sigma)$, and let $M(\sigma)$ be its dual lattice, which is embedded as a lattice in $N_{\mathbb{R}}$ via the inner product $*$. A unit simplex Δ with respect to the lattice $M(\sigma)$ is assigned $\operatorname{Vol}_{*,\sigma}(\Delta) = 1$. We caution that such normalization of the volume displays behavior that requires some care; for example, the in cone $\operatorname{cone}\{\mathbf{e}_{12}\}$ in $\mathbb{R}^2$ with the standard inner product, the line segment $[0, \mathbf{e}_{12}]$ has volume 2.

Theorem 7.4. [NR23, Main Result] Suppose Σ is balanced. For a $*$-pseudo-cubical divisor D on Σ, we have

$$\deg_\Sigma(D^d) = \operatorname{Vol}_{*,\Sigma}(P_{*,\Sigma}(D)) := \sum_{\sigma\in\Sigma_{\max}} \operatorname{Vol}_{*,\sigma}(P_{*,\sigma}(D)).$$

We now apply this theory to multi-permutohedral fans. Let $*$ be the inner product on N^π obtained as the product of inner product each $\mathbb{R}^{E_i}/\mathbb{R}\mathbf{e}_{E_i}$ such that $\overline{\mathbf{e}}_j$ is a unit vector for all $j \in E_i$. Note that $\overline{\mathbf{e}}_i$ and $\overline{\mathbf{e}}_j$ are orthonormal if $i, j \in E$ satisfies $\pi(i) \neq \pi(j)$. In particular, the inner product $*$ has the following pleasant feature.

Proposition 7.5. For a π-colored subset S, the set $\{\overline{\mathbf{e}}_i : i \in S\}$ is simultaneously an orthonormal basis of its linear span $\mathbb{R}^S$ and a $\mathbb{Z}$-basis of the lattice $\mathbb{R}^S \cap N^\pi_{\mathbb{R}}$. Thus, for every maximal cone σ of Σ^π, whose span is $\mathbb{R}^T$ for some π-transversal T, the volume $\operatorname{Vol}_{*,\sigma}$ agrees with the usual volume on $\mathbb{R}^T \simeq \mathbb{R}^n$ where the standard simplex has volume 1.

For the proof of Theorem A, we would like to compute $\deg_{\Sigma_\pi}\left((x_1 h_{S_1} + \cdots + x_m h_{S_m})^n\right)$ in this way, but $x_1 h_{S_1} + \cdots + x_m h_{S_m}$ is generally not pseudo-cubical even if all $x_1, \ldots, x_m$ are all nonnegative. We thus consider the following class of divisors that is slightly larger than the pseudo-cubical divisors.

Definition 7.6. A *multi-polymatroid* is a function $\mathrm{rk}_\pi : 2^\pi \to \mathbb{R}$ on the set of π-colored subsets of E such that for every π-transversal T, the restriction $\mathrm{rk}_\pi|_{2^T}$ is a polymatroid on T. The *independence polytopal complex* $IP(\mathrm{rk}_\pi)$ of a multi-polymatroid rk_π is the polyhedral complex in $N^\pi_\mathbb{R}$ defined as the union

$$IP(\mathrm{rk}_\pi) = \bigcup_{\pi\text{-colored } S} IP(\mathrm{rk}_\pi|_{2^S})$$

where each $IP(\mathrm{rk}_\pi|_{2^S})$ is considered as a subset of $\mathbb{R}^S$. We will often equate the function rk_π with the divisor $D = \sum_S \mathrm{rk}_\pi(S) x_S$ on Σ^π.

Note that the set of multi-polymatroid is a full-dimensional convex cone in $\mathbb{R}^{2^\pi \setminus \{\emptyset\}}$. Moreover, it is straightforward to verify that each h_S is a multi-polymatroid, whose independence polytopal complex satisfies $IP(h_S|_{2^T}) = \Delta^0_{S\cap T}$ for every π-transversal T.

Note that $IP(\mathrm{rk}_\pi)$ is a subset of the support of the multi-permutohedral fan. One may compare $IP(\mathrm{rk}_\pi)$ and the normal complex $P_{*,\Sigma^\pi}(D_{\mathrm{rk}_\pi})$.

Proposition 7.7. [CDE+24, Lemma 4.22 and 5.3] For a function rk_π, if the divisor $D = \sum_S \mathrm{rk}_\pi(S)$ on Σ^π is pseudo-cubical then D is a poly-multimatroid. Moreover, in that case we have $IP(\mathrm{rk}_\pi) = P_{*,\Sigma^\pi}(D)$.

Now, we come to an easy but crucial observation:

Lemma 7.8. The volume $\mathrm{Vol}_{*,\Sigma^\pi}(IP(\mathrm{rk}_\pi))$, as a function of rk_π on the full-dimensional cone of multi-polymatroids, is a polynomial. This polynomial equals the polynomial $\deg_{\Sigma^\pi}(D^n_{\mathrm{rk}_\pi})$ if there exists a nonempty open subset of cubical divisors D_{rk_π} for which the two polynomials agree.

Proof. The first statement follows from that the volume is the sum of the volumes of $IP(\mathrm{rk}_\pi|_{2^T})$ for each π-transversal T. The second statement follows from that the zero loci of a nonzero polynomial cannot be dense. □

Proof of Theorem A. By Theorem 2.5 and Proposition 6.5, it suffices to show that $\mathrm{Vol}_{*,\Sigma^\pi}(IP(\mathrm{rk}_\pi))$ equals $\deg_{\Sigma^\pi}(D^n_{\mathrm{rk}_\pi})$ for all multi-polymatroid rk_π. But these as polynomials in rk_π agree by the lemma and the proposition above. □

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