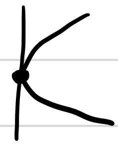
N.B. $X \subset \mathbb{P}^n$ of $\dim = d > 0$.
X ACM $\iff \Gamma_*(\mathcal{O}_X) = S_X$
& $H^i_*(\mathcal{O}_X) = 0 \ \forall 1 \le i < d$

$\boxed{\text{Intro. to Liaison Thry}}$ Chris Eur

N.B. $(R, \mathfrak{m})$ Noeth. local (often RLR) ① sheaf-theoretically (work locally)
$(S, \mathfrak{m})$, $S = k[x_0, \dots, x_n]_\bullet$, $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$ ② try same proof (may need ACM)

E.g. Twisted cubic $C = V(I_1)$, $I_1 = (x_0x_3 - x_1x_2,\ x_0x_2 - x_1^2,\ x_1x_3 - x_2^2)$.
$X = V(J)$, $J = (x_0x_3 \overset{f}{-} x_1x_2,\ x_0x_2 \overset{g}{-} x_1^2)$, $X = C \cup V(x_0, x_1)$, $I_2 = (x_0, x_1)$.
$(J : I_1) = I_2$, $(J : I_1) = I_2$



$\left(\begin{array}{l} x_0x_1x_3 - x_0x_2^2 = x_1 f - x_2 g \\ x_1^2x_3 - x_1x_2^2 = x_2 f - x_3 g \end{array}\right)$

Defn $V_1, V_2 \subset \mathbb{P}^n$ *algebraically linked* through a c.i. $X \subset \mathbb{P}^n$ if
① V_1, V_2 equidim. & no emb. comp., ② $\mathcal{I}(V_2)/\mathcal{I}(X) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{V_1}, \mathcal{O}_X)$
$\mathcal{I}(V_1)/\mathcal{I}(X) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{V_2}, \mathcal{O}_X)$

geometrically linked if: ① V_1, V_2 equidim, no emb. comp, no shared comp.
② $V_1 \cup V_2 = X$ (a c.i.).

E.g. (self-linkage) $V(x_0x_2 - x_1^2,\ x_2(x_1x_3 - x_2^2) - x_3(x_0x_3 - x_1x_2)) = C \cup C$.
(alg. linked to itself, but not geom.).

Question What kind of equiv. relation does alg./geom linkage generate?
Today: ① When is V linked (via a chain) to a c.i.?
② How are properties of linked V_1, V_2 related?

Prop Geom linked $\Rightarrow$ alg. linked.
pf) Check locally: $\mathfrak{a}/\mathfrak{a}\cap\mathfrak{b} \overset{?}{\xrightarrow{\sim}} \mathrm{Hom}(R/\mathfrak{b}, R/\mathfrak{a}\cap\mathfrak{b}) = (0 :_{R/\mathfrak{a}\cap\mathfrak{b}} \mathfrak{b})$.
Well, $(\mathfrak{a}\cap\mathfrak{b} :_R \mathfrak{b}) = (\mathfrak{a} :_R \mathfrak{a}+\mathfrak{b}) = \bigcap_i (Q_i :_R \mathfrak{a}+\mathfrak{b}) = \bigcap_i Q_i = \mathfrak{a}$, since
$V(\mathfrak{a}+\mathfrak{b})$ has codim ≥ 1 in $V(\mathfrak{a})$ and $(Q_i :_R I) = Q_i$ for $Q_i \not\supset I$
($\because$ $Q_i \in \mathrm{Ass}\, R/\mathfrak{a}$ so $Q_i \not\supset I \Rightarrow \exists r \in I$ nzd on R/Q_i).

N.B. $(x_1, \dots, x_r)$ a M-seq. in $\operatorname{Ann} N$. Then $\operatorname{Ext}^r(N, M) \simeq \operatorname{Hom}(N, M/(\underline{x}))$
pf)

N.B. V_1, V_2 generically c.i.
$\Uparrow$

Prop Alg. linked + no shared comp + (locally CM) $\Rightarrow$ geom. linked
pf) locally, say $X = \operatorname{Spec} R$, $\mathcal{I}_1/\mathcal{I}_X$, $\mathcal{I}_2/\mathcal{I}_X = \mathfrak{a}_1, \mathfrak{a}_2 \subset R$.
No shared comp $\Rightarrow$ $\operatorname{Supp}(\mathfrak{a}_1 \cap \mathfrak{a}_2)$ has codim ≥ 1, but if ϕ, then ※.

Lem $(R, \mathfrak{m})$ Gorenstein. $0 \neq \mathfrak{a}_1 \subset R$ st $\dim R = \dim R/\mathfrak{a}_1$, $\mathfrak{a}_2 := (0 : \mathfrak{a}_1)$.
Then $R/\mathfrak{a}_1$ CM $\Leftrightarrow$ $R/\mathfrak{a}_2$ CM and $R/\mathfrak{a}_1$ has no emb. comp.
In this case, $\mathfrak{a}_1 = (0 : \mathfrak{a}_2)$ and $\dim R/\mathfrak{a}_2 = \dim R$.

Rem $V_1 \subset \mathbb{P}^n$ equidim and $V_1 \subset X$ a c.i. Let $V_2 \subset \mathbb{P}^n$ be defined by $\mathcal{I}(V_2)/\mathcal{I}(X) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{V_1}, \mathcal{O}_X)$. Then
① V_2 alg. linked to V_1 if V_1 no emb. comp. & locally CM.
② V_1 is ACM $\Leftrightarrow$ V_2 is ACM & V_1 has no emb. comp.

E.g. $V(x_0x_3 - x_1x_2,\ x_0x_2^2 - x_1^2x_3) = \underset{=}{\tilde{C}}$ (twisted quartic) $\cup L_1 \cup L_2$
$V(\text{—''—}, x_1^3 - x_0^2x_2, x_2^3 - x_1x_3^2) = \{[1 : s : s^3 : s^4]\} \cup V(x_0, x_1) \cup V(x_2, x_3)$
Note : ① $\tilde{C}$, $L_1 \cup L_2$ both not ACM (but still alg. linked)
② $\tilde{C} \cup L_1$, L_2 both ACM
③ $\tilde{C} \cup L_1$ cannot be A. Gorenstein (if so, then $\tilde{C}$ wld be ACM).

pf Lem) $\boxed{\text{KEY}}$: local duality : $\operatorname{Ext}^i(M, \omega_R) \simeq \operatorname{Hom}(H_{\mathfrak{m}}^{d-i}(M), E(R/\mathfrak{m}))$
$(\operatorname{Hom}(R/\mathfrak{a}_1, R) = \mathfrak{a}_2)$. $\omega_R \cong R$ since Gor. $H_{\mathfrak{m}}^{d-i}(M) = 0$ for $i \geq 1$ if M is MCM.

Thm $V \subset \mathbb{P}^n$ codim $= 2$ (generic c.i.). Then TFAE: (i) V is ACM,
(ii) $\exists\, V_1, \dots, V_s$, $V_1 = V$, V_s a c.i. st V_i, V_{i+1} are alg. (geom.) linked.
Moreover, minimum such s is $\#\text{mingen}\, I(V) - 1$.

Proof of Thm) $\Leftarrow$: immediate since c.i. is (A)CM.

$\Rightarrow$: Prove in local setting: $(R, \mathfrak{m})$ RLR, $R/\mathfrak{a}_1$ CM.

<u>Claim</u>: Take $\bar{r} := (r_1, r_2)$ R-reg. seq. in $\mathfrak{a}_1$ extending to min. gen. of $\mathfrak{a}_1$.

Then $\mathfrak{a}_2 = \langle r_1, r_2 \rangle : \mathfrak{a}_1$ has $\mu(\mathfrak{a}_2) = \mu(\mathfrak{a}_1) - 1$.

$0 \to R^{n-1} \to R^n \to R \to R/\mathfrak{a}_2 \to 0$: apply $\mathrm{Hom}(-, R)$

$$0 \leftarrow \underset{\substack{\wr\wr \\ \mathrm{Hom}_{R/\bar{r}}(R/\mathfrak{a}_2, R/\bar{r}) \simeq \mathfrak{a}_1/\bar{r}}}{\mathrm{Ext}^2_R(R/\mathfrak{a}_2, R)} \leftarrow (R^{n-1})^\vee \leftarrow (R^n)^\vee \leftarrow \cdots$$

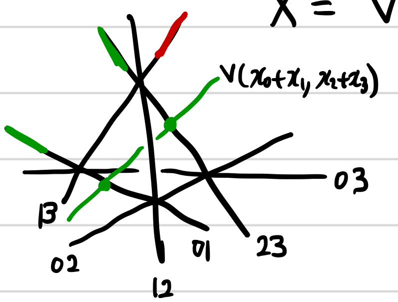
$\Rightarrow \mu(\mathfrak{a}_1) - 2 = \mu(\mathfrak{a}_2) - 1$ ✓

<u>E.g.</u> $\mathbb{P}^3 \supseteq V_1 = V(x_0x_1x_2, x_0x_1x_3, x_0x_2x_3, x_1x_2x_3) = \bigcup_{i<j} V(x_i, x_j)$ (6 coord. lines).

$X = V(x_0x_1(x_2+x_3), x_2x_3(x_0+x_1)) \Rightarrow V_2 = V(x_0x_3 + \underbrace{x_1x_3, x_1x_2}_{X'} + x_1x_3, x_0x_2 - x_1x_3)$.

$\quad = V_1 \cup V(x_0+x_1, x_2+x_3)$ w/ double 01, 23

$V_3 = V(x_1, x_3) \leftsquigarrow X'$



<u>E.g.</u> The degree seq. of Vs is not unique:

$I = I(\tilde{C} \cup L_1) = (x_0x_3 - \underbrace{x_1x_2, x_0x_2^2}_{(x_2, x_3)} - \underbrace{x_1^2x_3, x_1^3}_{(x_0x_2, x_1^2)} - x_0^2x_2)$

$\leadsto$ <u>Q.</u> What if we choose $\bar{r}$ w/ minimal degrees each time? Then Yes:
(the following answers this & more)

<u>Prop</u> R RLR, $\mathfrak{a}_1, \mathfrak{a}_2$ CM alg. linked thru $\langle \bar{r} \rangle$. $F_\bullet \to R/\mathfrak{a}_1$ free res, $K_\bullet \to R/\langle \bar{r} \rangle$ Koszul res., $K_\bullet \xrightarrow{\alpha} F_\bullet$ from $R/\langle \bar{r} \rangle \twoheadrightarrow R/\mathfrak{a}_1$. Then the mapping cone of $\alpha^\vee: F_\bullet^\vee \to K_\bullet^\vee$ gives resoln of $R/\mathfrak{a}_2$. (shift $\alpha^\vee$ by $-e = \sum_i \deg r_i$ in the graded case).

E.g. $V_1 = C$.

$$\begin{tikzcd} K_\bullet \arrow[d, "\alpha"] & 0 \arrow[r] & S(-4) \arrow[r] \arrow[d] & S(-2)^2 \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & S_X \arrow[r] \arrow[d, two heads] & 0 \\ F_\bullet & 0 \arrow[r] & S(-3)^2 \arrow[r] & S(-2)^3 \arrow[r] & S \arrow[r] & S_C \arrow[r] & 0 \end{tikzcd} \qquad (e=4)$$

$$\rightsquigarrow$$

$$\begin{tikzcd} S & S(-2)^2 \arrow[l] & S(-4) \arrow[l] \\ S(-1)^2 \arrow[u] & S(-2)^3 \arrow[l] \arrow[u] & S(-4) \arrow[l] \arrow[u] \end{tikzcd}$$

$$\rightsquigarrow$$

$$0 \leftarrow S_{V_2} \leftarrow S \leftarrow S(-1)^2 \leftarrow S(-2) \leftarrow 0$$

N.B. Can use this to recover: $p(C) - p(C') = \frac{1}{2}(\deg f + \deg g - 4)(d(C) - d(C'))$ for ACM curves C, C'! More, any ACM codim$=2$ curve is linked to $V(l_1, l_2)$.

pf Prop)

$$\begin{tikzcd} H_0(F_\bullet^\vee) \arrow[rr] & & H_0(K_\bullet^\vee) \\ \mathrm{Ext}_R^{\mathrm{codim}}(R/\mathfrak{a}_1, R) \arrow[u, phantom, "\cong"] \arrow[rr, "\mathrm{Ext}^{\mathrm{codim}}(\cdot, R)"] & & \mathrm{Ext}_R^{\mathrm{codim}}(R/F, R) \arrow[u, phantom, "\cong"] \\ \mathrm{Hom}_{R/F}(R/\mathfrak{a}_1, R/F) \arrow[u, phantom, "\cong"] & & \mathrm{Hom}_{R/F}(R/F, R/F) \arrow[u, phantom, "\cong"] \\ \mathfrak{a}_2/F \arrow[u, phantom, "\cong"] & & \end{tikzcd}$$

$$\begin{tikzcd} \arrow[d, dotted] & \arrow[d, dotted] \\ F_{d-1}^\vee \arrow[r] \arrow[d] & K_{d-1}^\vee \arrow[d] \\ F_d^\vee \arrow[r] \arrow[d] & K_d^\vee \arrow[d] \\ \mathfrak{a}_2/F \arrow[r] & R/F \end{tikzcd}$$

Thm Alg. linkage & geom. linkage generate the same equiv. class. (for subschemes of $\mathbb{P}^n$ equidim, no emb. comp, locally CM, gen. c.i.)

Defn Let $C \subset \mathbb{P}^3$ equidim, no emb. comp, locally CM, gen. c.i. curve. $M(C) := H^1_*(\mathcal{I}) = \bigoplus_m H^1(\mathcal{I}(m))$. (Hartshorne-Rao module)

N.B. For space curves, $M(C) = 0 \iff C$ is ACM.