COMPLEX ANALYSIS NOTES

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Notes taken while reviewing (but closer to relearning) complex analysis through [SSh03] and [Ahl79]. Some solutions to the exercises in [SSh03] are also written down. I do not claim that the notes or solutions written here are correct or elegant.

1. Preliminaries to complex analysis

The **complex numbers** is a field $\mathbb{C} := \{a + ib : a, b \in \mathbb{R}\}$ that is complete with respect to the modulus norm $|z| = z\overline{z}$. Every $z \in \mathbb{C}, z \neq 0$ can be uniquely represented as $z = re^{i\theta}$ for $r > 0, \theta \in [0, 2\pi)$. A **region** $\Omega \subset \mathbb{C}$ is a connected open subset; since $\mathbb{C}$ is locally-path connected, connected+open $\implies$ path-connected (in fact, PL-path-connected). Denote the open unit disk by $\mathbb{D}$.

Definition 1.1. *A function $f : U \to \mathbb{C}$ for $U \subset \mathbb{C}$ open is **holomorphic/analytic/complex-differentiable at** $z_0 \in U$ if*

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

*exists, and we denote the limit value by $f'(z_0)$. Equivalently, f is holomorphic at z_0 iff there exists $a \in \mathbb{C}$ and such that $f(z_0 + h) - f(z_0) - ah = h\psi(h)$ and $\psi(h) \to 0$ as $h \to 0$, in which case $a = f'(z_0)$. f is **holomorphic** if it is at z_0 for all $z_0 \in U$.*

Proposition 1.2. *Differentiation rules about $f + g, fg, f/g$ and $f \circ g$ (chain rule) holds.*

Theorem 1.3. *For $f : U \to \mathbb{C}$, write $f = u + iv$ where $u, v : U \to \mathbb{R}$. f is holomorphic at $z_0 \in U$ if and only if f as a map $\mathbb{R}^2 \supset U \to \mathbb{R}^2$ is differentiable at z_0 and satisfies*

Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$ at z_0

Proof. First, note that $a + ib \in \mathbb{C}$ can identified with the real matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. This also works well with $\mathbb{C} \simeq \mathbb{R}^2$ in that the vector in $\mathbb{R}^2$ for $(a + ib)(c + id)$ is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$.

Now, as a map in real variables, f is differentiable iff there exists a matrix A such that $|f(z_0 + h) - f(z_0) - Ah| = |h||\psi(h)|$ with $|\psi(h)| \to 0$ as $h \to 0$. Now, multiplication by A is complex number multiplication iff A of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Thus, if f is differentiable in real sense and satisfies the Cauchy-Riemann equations, then $f(z_0 + h) - f(z_0) - (u_x(z_0) + iv_x(z_0))h = h\psi(h)$ with $|\psi(h)| \to 0$ as $h \to 0$, and hence holomorphic at z_0. If f is holomorphic, then letting A be the matrix of $f'(z_0)$ works, and thus Cauchy-Riemann equation follows. □

Definition 1.4. *Define two differential operators by:*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Proposition 1.5. *f is holomorphic at z_0 iff $\dfrac{\partial f}{\partial \overline{z}}(z_0) = 0$. Moreover, if holomorphic,*

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0) = 2i\frac{\partial v}{\partial z}(z_0) \text{ and } \det[Df]_{z_0} = |f(z_0)|^2$$

Power series are good (and really the only) examples of holomorphic functions.

Theorem 1.6. *Given a power series $\sum_{n=0}^{\infty} a_n z^n$, let $1/R := \limsup |a_n|^{1/n}$ (with $1/\infty = 0$ and $1/0 = \infty$). Then for $|z| < R$, the series (uniformly) converges absolutely, and diverges for $|z| > R$. Moreover, $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on its disk of convergence with $f'(z) = \sum_{n=0}^{\infty} n a_n z^n$ with the same radius of convergence.*

Proof. Compare to geometric series (Weierstrass M-test), and do some computation. □

It is useful to note the relationship between the root-test and the ratio-test; ratio-test is often the easier option, but root-test is more general. More precisely,

Proposition 1.7. *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf \frac{c_{n+1}}{c_n} \leq \liminf \sqrt[n]{c_n} \quad \text{and} \quad \limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}$$

1.8. **Exercises.**

Exercise 1.A. [SSh03, 1.4] *Show that there is no total ordering on $\mathbb{C}$*

Proof. Suppose there is a total ordering $>$ on $\mathbb{C}$, and WLOG $i > 0$. Then $-1 = i^2 > 0$, and so $-1 > 0$, and so $1 = (-1)^2 > 0$ but $-1 + 1 > 1$. Thus, $1 > 0$ and $1 < 0$, which is a contradiction. □

Exercise 1.B. [SSh03, 1.7] *For $z, w \in \mathbb{C}$ such that $\overline{z}w \neq 1$ and $|z| \leq 1, |w| \leq 1$, show that*

$$\left| \frac{w - z}{1 - \overline{w}z} \right| \leq 1$$

where the equality occurs exactly when $|z| = 1$ or $|w| = 1$. Moreover, for $w \in \mathbb{D}$, the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$ is a bijective holomorphic map $F : \mathbb{D} \to \mathbb{D}$ that interchanges 0 and w, and $|F(z)| = 1$ if $|z| = 1$. These mappings are called ***Blaschke factors***

Proof. The inequality is equivalent to $|w - z|^2 \leq |1 - \overline{w}z|^2$, which when written out is equivalent to $|z|^2 + |w|^2 \leq 1 + |w|^2|z|^2$, and this inequality holds with equality exactly at $|z| = 1$ or $|w| = 1$ since $0 \leq (1 - |w|^2)(1 - |z|^2)$ for $|z|, |w| \leq 1$. One computes that $F \circ F(z) = z$ and the rest of claims about F follows immediately from the inequality. □

Exercise 1.C. [SSh03, 1.9] *Show that Cauchy-Riemann equations in polar coordinates is*

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta$$

Proof. With $x = r\cos\theta, y = r\sin\theta$, computing du for $u : \mathbb{R}^2 \to \mathbb{R}$ in two coordinates (x, y) and (r, θ) gives us (and likewise for dv):

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} u_r \\ u_\theta \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_r \\ v_\theta \end{bmatrix}$$

and thus we have

$$\begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_r & v_r \\ u_\theta & v_\theta \end{bmatrix} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$$

Now, $u_x = v_y$ and $u_y = -v_x$ becomes:

$$(1) : r\cos\theta u_r - \sin\theta u_\theta = r\sin\theta v_r + \cos\theta v_\theta, \quad (2) : r\sin\theta u_r + \cos\theta u_\theta = -r\cos\theta v_r + \sin\theta v_\theta$$

And from here $(1)\cdot\cos\theta + (2)\cdot\sin\theta$ gives us $ru_r = v_\theta$, and $-(1)\cdot\sin\theta + (2)\cdot\cos\theta$ gives us $rv_r = -u_\theta$, as desired. □

Exercise 1.D. [SSh03, 1.10,11] *Show that* $4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \Delta$ *where* Δ *is the* ***Laplacian*** $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. *Moreover, show that if* f *is holomorphic on an open set* Ω*, then real and imaginary parts of* f *are* ***harmonic****, i.e. Laplacian is zero.*

Proof. $4\frac{1}{2}(\partial_x - i\partial_y)\frac{1}{2}(\partial_x + i\partial_y) = \Delta$, and f holomorphic means $\frac{\partial f}{\partial \bar{z}} = 0$, and so $\Delta f = 0$. □

Exercise 1.E. [SSh03, 1.13] *If* f *is holomorphic on an open set* Ω*, and (i)* $\mathrm{Re}(f)$*, or (ii)* $\mathrm{Im}(f)$*, or (iii)* $|f|$ *is constant, then* f *is constant on* Ω*.*

Proof. It suffices to show that $f' = 0$ on Ω on any of the conditions given. For (i) or (ii), $\frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z} = i2\frac{\partial v}{\partial z}$, so $f' = 0$. For (iii), $u^2 + v^2$ is constant, and so applying $\partial_{xx}, \partial_{yy}$ to $(u^2 + v^2) = C$ gives us $u_{xx}u + v_{xx}v + (u_x^2 + v_x^2) = 0, u_{yy}u + v_{yy}v + (u_y^2 + v_y^2) = 0$. Adding the two and using the fact that u, v are harmonic, we have that $u_x = u_y = v_x = v_y = 0$. □

Exercise 1.F. [SSh03, 1.14,15] *Prove the* ***summation by parts*** *formula (defining* $B_k := \sum_{n=1}^{k} b_n$ *and* $B_0 := 0$*),*

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M b_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

and use the formula to prove the ***Abel's theorem****: If* $\sum_{n=1}^{\infty} a_n$ *converges, then*

$$\lim_{r \to 1^-} \sum_{n=1}^{\infty} a_n r^n = \sum_{n=1}^{\infty} a_n$$

Proof. For the summation by parts formula, draw the $n \times n$ matrix $(a_i b_j)_{1 \le i,j \le n}$ and consider what each terms in the summation mean. As for Abel's theorem, something is weird: since $f_N(r) = \sum_{n=1}^{N} a_n r^n$ is continuous on $0 \le r \le 1$ and $f_N \to f$ uniformly (where $f := \sum_{n=1}^{\infty} a_n r^n$), we can commute the two limits. □

Exercise 1.G. [SSh03, 1.20] *Show that: (1)* $\sum n z^n$ *diverges for all points on the unit circle, (2)* $\sum \frac{1}{n^2} z^n$ *converges for all points on the unit circle, (3)* $\sum \frac{1}{n} z^n$ *converges for all points on the unit circle except* $z = 1$*.*

Proof. For (1), each terms don't go to zero. For (2), absolute convergence. For (3), we need: **Lemma**: Suppose partial sums A_n of $\sum a_n$ is a bounded sequence, and $b_0 \ge b_1 \ge b_2 \ge \cdots$ with $\lim_{n\to\infty} b_n = 0$. Then $\sum a_n b_n$ is convergent. (Proof: use summation by parts formula).

This lemma also implies the Alternating Series Test with $a_n = (-1)^n$. For (3), we note that $a_n = z^n$ satisfies the condition of the lemma for $|z| \le 1, z \ne 1$. □

2. Cauchy's Theorem and Basic Applications

A curve γ is assumed piecewise differentiable unless otherwise noted. A curve γ is **closed** if the initial and end points are the same. A **R-path** is a curve entirely consisting of horizontal and vertical segments. Note that any region in $\mathbb{C}$ is R-path-connected.

A region Ω is **simply-connected** if $\pi_1(\Omega) = 0$, or equivalently, if any continuous map $S^1 \to \Omega$ extends to $B^2 \to \Omega$, or equivalently, if complement of Ω in $\widehat{\mathbb{C}}$ is connected.

2.1. Cauchy's Theorem.

Definition 2.2. *For $f : \Omega \to \mathbb{C}$ and $\gamma : I \to \Omega$, we define the **integral of f along** γ by:*

$$\int_\gamma f dz := \int_I f(\gamma(t))\gamma'(t)dt$$

Equivalently, the integral is the integration of a 1-form as follows: $\int_\gamma (udx - vdy) + i(udy + vdx)$.

Proposition 2.3. *The defining* $\text{length}(\gamma) := \int_\gamma |dz| = \int_I |\gamma'(t)|dt$*, one has the following inequality:*

$$\left|\int_\gamma f dz\right| \leq \int_\gamma |f||dz| \leq \left(\sup_\gamma |f|\right) \cdot \text{length}(\gamma)$$

Theorem 2.4. *For a 1-form $\omega = pdx + qdy$ on an open region Ω, $\int_\gamma pdx + qdy = 0$ for any closed curve γ in Ω if and only if ω is exact. Moreover, if $\omega = df$, then for any $\gamma : [a, b] \to \Omega$,*

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$

Proof. The second part is easy, and it implies one direction of the first part. For the converse, if the integral along any closed curve is zero, pick an arbitrary point $p \in \Omega$ and define $F(z) := \int_\gamma \omega$ for $z \in \Omega$ where γ is a curve from p to z. By making γ an R-path, with last segment being horizontal or vertical, one recovers that $dF = \omega$. □

Corollary 2.5. *If $f : \Omega \to \mathbb{C}$ has a **primitive**, i.e. $F : \Omega \to \mathbb{C}$ such that $F' = f$, then $\int_\gamma f = 0$ for all closed $\gamma \subset \Omega$.*

Proof. If $F = U + iV$ and $F' = f = u + iv$, then $dF = U_x dx + U_y dy + i(V_x dx + V_y dy)$ and $u = U_x = V_y, v = V_x = -U_y$, so that f as a 1-form equals dF. □

Theorem 2.6. *[Goursat's Theorem] If f is analytic on R, a rectangle with horizontal and vertical sides, then*

$$\int_{\partial R} f dz = 0$$

Proof. Keep subdividing rectangles into fours and pick ones with biggest integral and converge to z_0. At each step, we have $|\eta(R_n)| \geq 4^{-n}|\eta(R)|$. Now, make n large enough (R_n small enough to z_0) so that

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| < \epsilon|z - z_0|$$

Note that $\int_{\partial R} dz = 0 = \int_{\partial R} zdz$, so integrating both sides of inequality above gives $|\eta(R_n)| \leq e\int_{\partial R_n} |z - z_0||dz|$. Rest is computation. □

Proposition 2.7. *Theorem 2.6 still holds if f is holomorphic on $R\backslash\{z_1, \ldots, z_k\}$ ($z_i \in \text{int}(R)$) where*

$$\lim_{z \to z_i} (z - z_i)f(z) = 0 \quad \forall i$$

Proof. WLOG let $k = 1$ and use Theorem 2.6 to shrink the boundary of rectangle to a very small square centered at z_1. □

Theorem 2.8. *[Cauchy's Theorem I] If f is holomorphic on an open disk D (or on D minus finite points satisfying the condition in Proposition 2.7), then for any closed $\gamma \subset D$,*

$$\int_\gamma f dz = 0$$

Proof. Construct the primitive of f as $F(z) := \int_\sigma f dz$ where σ is an R-path from a pre-fixed point p to z. F is well-defined due to Theorem 2.6 (Proposition 2.7). □

Theorem 2.9. *Suppose f is holomorphic on open region Ω. Then if $\gamma_0, \gamma_1 \subset \Omega$ are homotopic (need be end-point homotopy), then*

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz$$

Proof. Let $\gamma_s(t) : I \times I \to \Omega$ be the homotopy. Since $\mathrm{Im}(\gamma_s(t)) \subset \Omega$ is compact, there exists $\epsilon > 0$ such that any 3ϵ-ball around a point in $\mathrm{Im}(\gamma_s(t))$ is contained in Ω. Also, there exist $\delta > 0$ such that $\sup |\gamma_{s_0}(t) - \gamma_{s_1}(t)| < \epsilon$ whenever $|s_0 - s_1| < \delta$. Use these to make disks $\{D_0, \ldots, D_n\}$ of radius 2ϵ, and consecutive points $\{z_0, \ldots, z_{n+1}\} \subset \gamma_{s_0}, \{w_0, \ldots, w_{n+1}\} \subset \gamma_{s_1}$ with $z_0 = w_0,\ z_{n+1} = w_{n+1}$ such that $z_i, z_{i+1}, w_i, w_{i+1} \in D_i$. Now Theorem 2.8 integrals implies that integrals along closed curves $z_i \overset{\text{straight}}{\to} w_i \overset{\gamma_{s_1}}{\to} w_{i+1} \overset{\text{straight}}{\to} z_{i+1} \overset{-\gamma_{s_0}}{\to} z_i$ is zero, and adding these up we have $\int_{-\gamma_{s_0}+\gamma_{s_1}} f dz = 0$. To finish the proof, divide interval I into many pieces all of length less than δ. □

Theorem 2.10 (Cauchy's Theorem II). *If f is holomorphic on an simply-connected region Ω, then for any closed $\gamma \subset \Omega$,*

$$\int_\gamma f = 0$$

Proof. Homotope γ to a constant map and use Theorem 2.9 □

Proposition 2.11. *Let $a \in \mathbb{C}$ and γ a closed curve not going through a. Then the **index of a point** a **with respect to** γ (or, the **winding number of** γ **around** a), defined as*

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a} dz$$

is an integer. In fact, if $C(\mathbb{C}\backslash\{a\})$ is the group of chains of closed curves in $\mathbb{C}\backslash\{a\}$, then the map $C(\mathbb{C}\backslash\{a\}) \to \mathbb{Z}$ given by $\gamma \mapsto n(\gamma, a)$ is the map $C(\mathbb{C}\backslash\{a\}) \to H_1(\mathbb{C}\backslash\{a\}) \xrightarrow{\sim} \mathbb{Z}$.

Proof. Homotope γ to lie on a circle centered at a and compute. For the second statement, note that H_1 is the abelianization of π_1. □

Proposition 2.12. *Given a closed curve γ, define the **regions determined by** γ as the connected open components of $\mathbb{C} - \gamma$. Then the number $n(\gamma, a)$ only depends on the region determined by γ that a belongs to.*

Proposition 2.13. *Let $C(\Omega)$ be the group of chains of closed curves on open region Ω. Given $\gamma \in C(\Omega)$, we have that $[\gamma] = 0 \in H_1(\Omega)$ if and only if $n(\gamma, a) = 0$ for any $a \in \mathbb{C} - \Omega$.*

Theorem 2.14 (General Cauchy's Theorem). *If f is holomorphic on an open region Ω, then*

$$\int_\gamma f dz = 0$$

for all $\gamma \in C(\Omega)$ such that $[\gamma] = 0 \in H_1(\Omega)$.

Proof. TODO □

2.15. **Basic Applications of Cauchy's Theorem.**

Remark 2.16. Even before touching upon calculus of residues, one can compute many real integrals using toy-contours and Cauchy's Theorem. (Examples in the Exercises)

Theorem 2.17 (Cauchy integral formulas). *Let f be holomorphic on a region Ω, and $\overline{D} \subset \Omega$ be a closed disk and $C := \partial\overline{D}$. Then for any $z \in D$,*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

Furthermore, one has that

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Proof. Fix $z_0 \in D$. By Theorem 2.8 on $\int_C \frac{f(\zeta)-f(z_0)}{\zeta - z_0} d\zeta = 0$, and linearity of integral gives $\int_C \frac{f(\zeta)}{\zeta - z_0} d\zeta = n(C, z_0) \cdot f(z_0)$. The second part of the theorem follows from the following more general lemma:

Lemma 2.18. [Ahl79, 4.2.3] *If $\phi(\zeta)$ is continuous on an arc γ, then $F_n(z) := \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n} d\zeta$ is holomorphic in each region determined by γ and $F_n'(z) = nF_{n+1}(z)$.*

□

Theorem 2.19 (General Cauchy's formula). *Let f be holomorphic on a region Ω, and γ be a cycle such that $\gamma \sim 0 \in H_1(\Omega)$. Then for any $z \in \Omega$ not on γ, we have*

$$n(\gamma, z) f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

Corollary 2.20 (Cauchy's inequality). *If f holomorphic on open Ω and $\overline{D}_R(z_0) \subset \Omega$, then*

$$|f^{(n)}(z_0)| \leq \frac{n! \|f\|_C}{R^n}$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$.

Theorem 2.21 (Morera's Theorem). *If f is continuous on open Ω and $\int_\gamma f dz = 0$ for all closed $\gamma \subset \Omega$, then f is holomorphic on Ω.*

Proof. Can define a primitive of f by $F(z) := \int_\sigma f dz$, and Theorem 2.17 implies that $F' = f$ is holomorphic as well. □

Remark 2.22. In the above statement, since any open set can be covered by open disks, it suffices to check $\int_{\partial R} f dz = 0$ for every rectangle $R \subset \Omega$.

Theorem 2.23 (Taylor's Theorem I). *Suppose f is holomorphic on a region Ω, and $\overline{D}_R(z_0) \subset \Omega$. Then for all $z \in D$, f has a power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!}$$

Proof. Let $C = \partial\overline{D}$ and by Theorem 2.17 write $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta$. Now, note that for any $|z - z_0| < r$ with $r < R$, we have a uniformly convergence series

$$\sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n = \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = (\zeta - z_0)\frac{1}{\zeta - z}$$

Uniform convergence means that we can interchange integral and the summation, and hence

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta\cdot(z-z_0)^n=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

□

Corollary 2.24. *Suppose f is holomorphic on $D_R(z_0)$. Then in the power series expansion $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ of f at z_0, the coefficients a_n are given by*

$$a_n=\frac{1}{2\pi r^n}\int_0^{2\pi}f(z_0+re^{i\theta})e^{-in\theta}d\theta$$

for any $0<r<R$.

Proof. Combine Theorem 2.17 and Theorem 2.23. □

Corollary 2.25 (Mean-value property)**.** *If f is holomorphic on $D_R(z_0)$, and $\mathrm{Re}(f)=u$, then*

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi}f(z_0+re^{i\theta})d\theta\quad\text{and}\quad u(z_0)=\frac{1}{2\pi}\int_0^{2\pi}u(z_0+re^{i\theta})d\theta$$

Theorem 2.26 (Analytic continuation)**.** *If f,g are analytic on a region Ω and agrees on a set with a limit point in Ω, then $f\equiv g$. (If $f=g$ on some open subset of Ω, then $f\equiv g$).*

Proof. One shows that zeroes of non-zero analytic functions are isolated by using Theorem 2.23 as follows: let E_1 be points where all derivatives vanish, and E_2 be points where at least one derivative is nonzero; both are open. □

Theorem 2.27 (Liouville's Theorem)**.** *If f is entire and bounded, then f is constant.*

Proof. Show $f'=0$ on any $z_0\in\mathbb{C}$ by Cauchy's inequality. □

Corollary 2.28 (Fundamental Theorem of Algebra)**.** *A polynomial $P(z)$ has a root in $\mathbb{C}$.*

Theorem 2.29. *If $\{f_n\}$ is holomorphic on a region Ω and $f_n\to f$ uniformly on every compact subset of Ω, then f is holomorphic on Ω. Moreover, $f_n'\to f'$ uniformly on every compact subset of Ω.*

Proof. Use uniform convergence to interchange limit and integral to find that f satisfies Morera's Theorem. For the second part, prove for every closed disk. □

Often a holomorphic function is thus built as $\sum_{n=0}^{\infty}f_n(z)$. (e.g. Zeta function). The following is the continuous version:

Proposition 2.30. *For an open Ω, suppose $F:\Omega\times[0,1]\to\mathbb{C}$ be continuous and $F(z,s)$ is holomorphic for each $s\in[0,1]$. Then $f(z):=\int_0^1F(z,s)ds$ is holomorphic.*

Let Ω be a symmetric open subset, in the sense that $z\in\Omega\Leftrightarrow\overline{z}\in\Omega$ (i.e. symmetric across the real-axis). In this case Ω partitions into Ω^+,Ω^-,I, the upper, lower, real-line parts of Ω. The next two theorems are in this setting.

Theorem 2.31 (Symmetry principle)**.** *If f^+ and f^- are holomorphic on Ω^+,Ω^-, and extends continuously to I with $f^+(x)=f^-(x)$ for all $x\in I$, then f defined piecewise accordingly on Ω is holomorphic.*

Proof. At each open disk in Ω centered on a point on I, use Morera with ϵ-shifting and partitions of rectangles under consideration. □

Theorem 2.32 (Schwarz reflection principle)**.** *Suppose f is holomorphic on Ω^+ and extends continuously to I with $f(I) \subset \mathbb{R}$. Then there exist F holomorphic on Ω such that $F = f$ on Ω^+.*

Proof. Define the lower half to be $F(z) = \overline{f(\overline{z})}$, and use the symmetry principle. □

2.33. **Exercises.**

Exercise 2.A. [SSh03, 2.1,2] *Evaluate the following integrals:*

$$\textbf{\textit{Fresnel integrals}}: \int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Proof. Follow the hint. □

Exercise 2.B. [SSh03, 2.7] *Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic, and let $d :=$ $\operatorname{diam}(f(\mathbb{D})) =$ $\sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$. Then*

$$2|f'(0)| \leq d$$

and equality holds precisely when f is linear.

Proof. For any $0 < r < 1$, we have that $2f'(0) = \frac{1}{2\pi i}\int_{\partial D_r} \frac{f(\zeta)-f(-\zeta)}{\zeta^2} d\zeta$, and thus $2|f'(0)| \leq \frac{1}{2\pi}\frac{d}{r^2}(2\pi r) = d/r$ for any $0 < r < 1$. Hence, $2|f'(0)| \leq d$, as desired. That equality holds when f is linear is clear. For converse, we first consider the following lemma:
Lemma: If f is holomorphic on $\mathbb{D}$ and non-constant, then $\exists z \in \mathbb{D}$ such that $|f(0)| < |f(z)|$. (Proof: If $f(0) = 0$ where is nothing to prove. So assume not can let $R > 0$ be such that $f(z) \neq 0$ on $|z| < R$. Note that for by Cauchy integral formula we have $|f(0)| \leq \frac{1}{2\pi}\int_{\partial D_r} \frac{|f(\zeta)|}{r}|d\zeta|$ for any $0 < r < R$. If $|f(0)| = \sup_{|\zeta|=r} |f(\zeta)|$, then $|f(\zeta)| = |f(0)|$ constant, so that f is constant by [SSh03, 2.15]. Thus, $\||f(0)| < \sup_{|\zeta|=r} |f(\zeta)|$.
Back to the main proof: now, use power series expansion and consider $f(z) - f(-z)$ to conclude that if reserved for later. turns out this is a hard problem □

Exercise 2.C. [SSh03, 2.12] *Let $u : \mathbb{D} \to \mathbb{R}$ be C^2 and harmonic (i.e. $\Delta u = 0$). Then show that there exists holomorphic f on $\mathbb{D}$ such that* $\operatorname{Re}(f) = u$*. Moreover, the imaginary part of f is unique upto a (real) additive constant.*

Proof. First, let $g(z) := 2\frac{\partial u}{\partial z}$. Note that g is holomorphic on $\mathbb{D}$ since $\frac{\partial g}{\partial \overline{z}} = 2\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}u = \frac{1}{2}\Delta u = 0$. By Cauchy's Theorem there exists F, unique upto (complex) additive constant, such that $F' = g$. So, writing $F = U + iV + c$ (where $c \in \mathbb{C}$), that $(F - u)' = 0$ implies that $(U - u)_x = (U_u)_y = 0$, and thus $U - u = \alpha$ for some $\alpha \in \mathbb{R}$. Absorbing this into c, we have constructed $f = F = u + iV + c$ where c is imaginary. □

Exercise 2.D. [SSh03, 2.13] *If f is holomorphic on a region Ω and for each $z_0 \in \Omega$ at least one coefficient in the power series expansion $f(z) = \sum_{n=0}^\infty c_n(z - z_0)^n$ is zero. Then show that f is a polynomial.*

Proof. Define $S_n := \{z \in \Omega : f^{(n)}(z) = 0\}$. Since $\bigcup_{n\in\mathbb{N}} S_n = \Omega$, there is N such that S_N is uncountable. Thus, $f^{(N)}(z)$ has zeroes that accumulate, and hence is identically zero. □

Exercise 2.E. [SSh03, 2.15] *Suppose f is continuous and non-zero on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$ such that $|f(z)| = 1$ for all $|z| = 1$. Show that f is then constant.*

Proof. Note that for any g holomorphic on $U \subset \mathbb{C}$ open, if $\phi : \mathbb{C} \to \mathbb{C}$ is the conjugation map, then $\widetilde{g}(z) := \overline{f(\overline{z})}$ is holomorphic on $\phi^{-1}(U)$. Thus, we can extend f to $|z| > 1$ by defining $f(z) := \frac{1}{\overline{f(\frac{1}{\overline{z}})}}$ (that $|f| = 1$ at $|z| = 1$ condition implies that the two f's match at $|z| = 1$). Now, using Morera's theorem with rectangles (and continuity of f), we have that f is entire, and since f was non-zero on $\overline{\mathbb{D}}$, f is bounded. By Liouville's theorem, f is thus constant. $\square$

3. Meromorphic Functions and the Logarithm

3.1. **Zeroes, singularities, meromorphic functions.**

Definition 3.2. *A point $z_0 \in \mathbb{C}$ is a **(point/isolated singularity** of f if f is defined in a neighborhood of z_0 but not at z_0.*

There are three types of point singularities: removable, poles, and essential singularities.

Theorem 3.3. *Suppose f is analytic on $\Omega\backslash\{z_0\}$. Then f can be extended to analytic function on Ω if and only if $\lim_{z\to z_0}(z-z_0)f(z) = 0$ (i.e. f is bounded on a neighborhood of z_0), and the extension is unique.*

Proof. By Proposition 2.7, we have that $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$ is valid for $z \neq z_0$ for a circle $C \subset \Omega$ centered at z_0, but the RHS expression is analytic inside the circle by Lemma 2.18, so extend f as the integral formula expresses. □

As a result of this theorem, isolated singularities that satisfy the condition in Theorem 3.3 are called **removable singularities**.

Theorem 3.4 (Taylor's Theorem II)**.** *If f is analytic on a region $\Omega \ni z_0$, then it is possible to write*

$$f(z) = \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right) + f_n(z)(z-z_0)^n$$

where f_n is analytic on Ω.

Proof. Apply Theorem 3.3 to $F(z) = \frac{f(z)-f(z_0)}{z-z_0}$ for case $n = 1$, and induct using the same idea. □

Theorem 3.5. *If f is analytic on a region Ω, does not vanish identically on Ω, and $f(z_0) = 0$, then there exists $g(z)$ analytic on Ω and nonzero in a neighborhood of z_0, and a unique n, such that*

$$f(z) = (z-z_0)^n g(z)$$

*(in which case, we say z_0 is a **zero of order** n).*

Definition 3.6. *A function f has a **pole** at z_0 if $1/f$, defined to be 0 at z_0, is analytic in a neighborhood of z_0. Equivalently, z_0 is a pole of f if $\lim_{z\to z_0} f(z) = \infty$.*

Theorem 3.7. *If f has a pole at z_0, then there exists h holomorphic and nonzero on a neighborhood of z_0, and a unique n, such that*

$$f(z) = (z-z_0)^{-n}h(z)$$

*(in which case, z_0 is a pole of **order/multiplicity** n).*

Corollary 3.8. *If f has a pole of order n at z_0, then*

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + G(z)$$

where $G(z)$ is holomorphic on a neighborhood of z_0.

Theorem 3.9 (Casorati-Weierstrass)**.** *Suppose f is holomorphic on a neighborhood of z_0 but not on z_0, which is an **essential singularity** (point singularity that is neither removable or a pole). Then the image of any (punctured) neighborhood of z_0 under f is dense in $\mathbb{C}$.*

Proof. Let D be a small disk around z_0, and suppose there exists w with $r > 0$ such that $D_r(w) \cap f(D) = \emptyset$. Now, consider the function $g(z) := \frac{1}{f(z)-w}$. Note that $g(z)$ is bounded on D, and hence has a removable singularity at z_0. If $g(z_0) \neq 0$, then f has removable singularity at z_0, and if $g(z_0) = 0$, then $f(z) - w$ has a pole at z_0, which means $f(z)$ has a pole at z_0. Either case, we get a contradiction. □

Definition 3.10. *If f is holomorphic on an unbounded region, we say that f has a* ***removable/pole/essential singularity*** *at ∞ if $F(z) := f(1/z)$ has the corresponding singularity at $z = 0$.*

Definition 3.11. *A function f is* ***meromorphic*** *on an open set Ω if it is holomorphic on Ω except for a discrete set of points which are poles of f.*

Theorem 3.12. *The meromorphic functions on $\widehat{\mathbb{C}}$ are the rational functions.*

Proof. Given f meromorphic on $\widehat{\mathbb{C}}$, subtract off principal part of f at each poles to get a bounded holomorphic function on $\mathbb{C}$, which must be constant. □

3.13. **The calculus of residues.**

Definition 3.14. *Suppose f has a pole of order n at z_0, so that by Corollary 3.8 we can write $f(z) = \dfrac{a_{-n}}{(z-z_0)^n} + \cdots + \dfrac{a_{-1}}{z-z_0} + G(z)$. We call the $\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}$ part the* ***principal part*** *of f at pole z_0, and define the* ***residue*** *of f at pole z_0 as $\operatorname{Res}_{z_0} f := a_{-1}$.*

Proposition 3.15. *If f has a pole of order n at z_0, then*

$$\operatorname{Res}_{z_0} f = \lim_{z\to z_0} \frac{1}{(n-1)!} \left(\frac{\partial}{\partial z}\right)^{n-1} (z-z_0)^n f(z)$$

Theorem 3.16 (Residue formula)**.** *Let f be analytic on a region Ω except for poles $z_1, \ldots, z_N \in \Omega$. Then, for any cycle $\gamma \sim 0 \in H_1(\Omega)$ and not passing through any of z_j's, we have*

$$\frac{1}{2\pi i} \int_\gamma f dz = \sum_{j=1}^{N} n(\gamma, z_j) \operatorname{Res}_{z_j} f$$

In particular, if γ is a toy-contour in Ω containing $z_1, \ldots, z_N$, then we have

$$\int_\gamma f dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{z_j} f$$

Proof. Note that $\gamma \sim 0$ in Ω implies that $\gamma \sim \sum_{j=1}^N n(\gamma, z_j) C_j$ in $\Omega \backslash z_1, \ldots, z_N$ for some circles C_j centered at z_j. For each C_j use Corollary 3.8. □

Example 3.17. [Ahl79, 4.5.3] *One can show (in increasing generalities) that for a rational function $R(x)$ such that $R(\infty) = 0$ and poles on the real line are simple, we get*

$$\int_{-\infty}^{\infty} R(x) e^{ix} = 2\pi i \sum_{y>0} \operatorname{Res}_y R(z) e^{iz} + \pi i \sum_{y=0} \operatorname{Res}_y R(z) e^{iz}$$

3.18. **The argument principle & applications.**

Theorem 3.19 (Argument principle)**.** *Suppose f is meromorphic on an open Ω with zeroes $\{a_j\}$ and poles $\{b_k\}$ (repeated to each order), and γ is a cycle such that $\gamma \sim 0 \in H_1(\Omega)$ and does not go through zeroes or poles of f. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

Proof. Apply the residue formula (Theorem 3.16) to f'/f. □

Corollary 3.20. *If f is meromorphic on an open set containing a circle C and its interior, and f has no zeroes or poles on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = (\text{number of zeroes inside } C) - (\text{number of poles inside } C)$$

where the zeros and poles are counted with multiplicity.

Theorem 3.21 (Rouche's theorem)**.** *If f and g are holomorphic on an open set containing a circle C and its interior, and $|f(z)| > |g(z)|$ for all $z \in C$, then f and $f+g$ have the same number of zeros in C.*

Proof. Define $f_t(z) = f(z) + tg(z)$ for $t \in [0,1]$, which is continuous jointly in t, z. Note that $|f(z)| > |g(z)|$ implies that $f_t(z) \neq 0$ for all t in a neighborhood of C. Thus, we can define

$$n_t := \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}dz$$

Since n_t is continuous in t, it must be constant, and hence $n_0 = n_1$, as desired. □

Theorem 3.22 (open mapping theorem)**.** *If f is holomorphic and non-constant on Ω, then f is open.*

Proof. Fix arbitrary z_0 and let $w_0 := f(z_0)$. Choose $\delta > 0$ such that $B_\delta(z_0) \subset \Omega$ and $f(z) \neq w_0$ on $|z - z_0| = \delta$, and $\epsilon > 0$ such that $|f(z) - w_0| \geq \epsilon$ on $|z - z_0| = \delta$. Now, note that for any w such that $|w - w_0| < \epsilon$, by Rouche's theorem we have that $g(z) := f(z) - w = (f(z) - w_0) + (w_0 - w) = F(z) + G(z)$ has a root in $|z - z_0| < \delta$. □

Theorem 3.23 (maximum modulus principle)**.** *If f is holomorphic and non-constant on a region Ω, then f cannot attain a maximum (i.e. maximum in modulus $|f(z)|$) in Ω.*

Proof. If $|f(z_0)|$ is max, then consider $f(D)$ where D is a small disk around z_0, which is open. □

Corollary 3.24. *Suppose Ω is a region with compact closure $\overline{\Omega}$. If f is holomorphic on Ω and continuous on $\overline{\Omega}$, then*

$$\sup_{z\in\Omega} |f(z)| \leq \sup_{\overline{\Omega}-\Omega} |f(z)|$$

3.25. **Complex logarithm.**

Proposition 3.26. *Suppose Ω is simply connected with $1 \in \Omega$ and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log z$ such that F is holomorphic on Ω, $e^{F(z)} = z$ for all $z \in \Omega$, and $F(r) = \log r$ whenever r is real number near 1.*

Example 3.27. In the split plane $\Omega = \mathbb{C} - \{(-\infty, 0]\}$, we have the **principal branch** $\log z = \log r + i\theta$ where $|\theta| < \pi$. For $\alpha \in \mathbb{C}$, z^α is defined as $z^\alpha := e^{\alpha \log z}$ on Ω

Theorem 3.28. *If f is nowhere vanishing holomorphic on simply connected region Ω, then there exists g holomorphic on Ω such that*

$$f(z) = e^{g(z)}$$

(i.e. $g(z) = \log f(z)$).

Proof. Fixing $z_0 \in \Omega$, define $g(z) = \int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta + c_0$ for γ path from z_0 to z and $e^{c_0} = f(z_0)$. □

3.29. **Exercises.**

Exercise 3.A. [SSh03, 3.1] *Show that the complex zeros of* $\sin \pi z$ *are exactly at the integers, and are each of order 1. Calculate the residue of* $1/\sin \pi x$ *are* $z = n \in \mathbb{Z}$.

Solution. Since $\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$, we have that $\sin \pi x = 0 \implies e^{i2\pi z} = 1$, and writing $z = x + iy$, one obtains $e^{i2\pi x} e^{-2\pi y} = 1$, so that $y = 0$ and $x = n \in \mathbb{Z}$. Power series expanding $\sin \pi z$ at $n \in \mathbb{Z}$ gives $\sum_{k=1}^{\infty} \pi(z-n) - \frac{\pi^3}{3!}(z-n)^3 + \cdots$ if n is even, and the opposite if n is odd. Hence, the zeros are of order 1, and the residues for $1/\sin \pi z$ are $1/\pi$ for n even and $-1/\pi$ for n odd. □

Exercise 3.B. [SSh03, 3.6] *Show that for* $n \geq 1$,

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{(2n)!}{4^n (n!)^2} \pi$$

Proof. Note that $f(z) := \frac{1}{(1+z^2)^{n+1}}$ has poles i and $-i$ of order $n+1$. So, [above integral equals $2\pi i \operatorname{Res}_i f = \lim_{z \to i} \frac{1}{n!} (\frac{\partial}{\partial z})^n \frac{(z-i)^{n+1}}{(1+z^2)^{n+1}} = 2\pi i \frac{(2n)!}{(n!)^2} \frac{(-1)^n}{(2i)^{2n+1}} = \frac{(2n)!}{4^n (n!)^2} \pi$. □

Exercise 3.C. [SSh03, 3.8] *Prove that*

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Proof. Letting $z = e^{i\theta}$, we can rewrite the integral as (where C is unit circle)

$$\int_C \frac{1}{a + b \cdot \frac{1}{2}(z + \frac{1}{z})} \frac{dz}{iz} = 2\pi i \operatorname{Res}_{z_0 \in \mathbb{D}} f$$

which gives us the desired result. □

Exercise 3.D. [SSh03, 3.10] *Show that for* $a > 0$,

$$\int_0^{\infty} \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a$$

Proof. Define $\log z$ on $\mathbb{C} - \{(0, y) : y \leq 0\}$ by $\log z = \log |z| + i\theta$ where $\theta \in (-\pi/2, 3\pi/2)$. Using the dented semicircle γ as the contour, and noting that $\frac{r \log r}{r^2 + a^2} \to 0$ as $r \to 0$ or $r \to \infty$, one computes that

$$2\pi i \cdot \frac{\log(ia)}{2ia} = \int_\gamma \frac{\log z}{z^2 + a^2} dz = \int_{-\infty}^{0} \frac{\log(-x) + i\pi}{x^2 + a^2} dx + \int_0^{\infty} \frac{\log x}{x^2 + a^2} dx$$

and thus we have $\frac{\pi \log a}{a} + \frac{i\pi^2}{2a} = 2 \int_0^{\infty} \frac{\log x}{x^2 + a^2} + \frac{i\pi^2}{2a}$, and the desired equality follows. □

Exercise 3.E. [SSh03, 3.14] *Prove that all entire functions that are also injective take the form* $f(z) = az + b$ *with* $a, b \in \mathbb{C}$ *and* $a \neq 0$.

Proof. If f is meromorphic on $\widehat{\mathbb{C}}$, then f is a rational function, but since f entire, it is a polynomial and injectivity implies that f is then linear. If f has essential singularity at infinity, then $f(\mathbb{C} \backslash \mathbb{D})$ must be dense in $\mathbb{C}$, but then since f is an open map, $f(\mathbb{C} \backslash \mathbb{D}) \cap f(\mathbb{D}) \neq \emptyset$, and hence injectivity implies that f cannot have essential singularity at infinity. □

Exercise 3.F. [SSh03, 3.15] *Prove the following statements:*

(1) *If* f *is an entire function satisfying* $\sup_{|z|=R} |f(z)| \leq AR^k + B$ *for some* $A, B \geq 0$ *and* $k \in \mathbb{N}$, *then* f *is polynomial of degree* $\leq k$.
(2) *If* f *is holomorphic on* $\mathbb{D}$, *is bounded, and converges uniformly to zero in the sector* $\theta < \arg z < \phi$ *as* $|z| \to 1$, *then* $f = 0$.
(3) *Let* $w_1, \ldots, w_n$ *be on the unit circle* C. *Then* $\exists z \in C$ *such that* $|z - w_1| \cdots |z - w_n| = 1$.

(4) *If the real part of an entire function f is bounded, then f is constant.*

Proof.

(1) Cauchy inequality implies that $f^{(n)}(0) = 0$ for all $n > k$.
(2) $\boxed{\text{ASK}}$
(3) Note that $C \to \mathbb{R}$ given by $z \mapsto |z - w_1| \cdots |z - w_n|$ is continuous, so it suffices to show that for some $z \in C$, $|z - w_1| \cdots |z - w_n| \geq 1$. Well, $(z - w_1) \cdots (z - w_n)$ is holomorphic on $\mathbb{D}$, then achieves modulus 1 when $z = 0$, so the maximum principle gives us the desired $z \in C$.
(4) If f has essential singularity at infinity, then the real part is not bounded by Casorati-Weierstrass. But if f is meromorphic, then f is a polynomial and hence is constant.

□

Exercise 3.G. [SSh03, 3.16] *Suppose f and g are holomorphic on a region containing $\overline{\mathbb{D}}$, and suppose f has a simple zero at $z = 0$ with no other zeroes on $\overline{\mathbb{D}}$. Then $f_\epsilon(z) = f(z) + \epsilon g(z)$ has a unique zero in $\overline{\mathbb{D}}$ for ϵ sufficiently small, and if z_ϵ is the zero of f_ϵ, then $\epsilon \mapsto z_\epsilon$ is continuous.*

Proof. For a small enough $\epsilon > 0$, we have $\inf_{|z|=1} |f| > \epsilon \sup_{|z|=1} |g|$, so that by Rouche's theorem f_ϵ has a unique zero in $\overline{\mathbb{D}}$. Moreover, let $\{\delta_n\}$ sequence of numbers converging to $\delta < \epsilon$. We need show that $z_{\delta_n} \to z_\delta$. Well, if $\{z_{\delta_n}\} \subset \overline{\mathbb{D}}$ does not converge to z_δ then it has a subsequence that converges to some $w \neq z_\delta$. But since $F : \overline{\mathbb{D}} \times \mathbb{R} \to \mathbb{C}$ defined as $F(z, \epsilon) := f_\epsilon(z)$ is continuous, and $(\delta_n, z_{\delta_n}) \to (\delta, w)$, we have $f_\delta(w) = F(w, \delta) = 0$, which contradicts uniqueness of the zero of f_δ. □

Exercise 3.H. [SSh03, 3.17] *Let f be non-constant and holomorphic on an open set containing $\overline{\mathbb{D}}$. Iff $|f(z)| = 1$ on $|z| = 1$, or if $|f(z)| \geq 1$ on $|z| = 1$ and there exists $z_\in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disk.*

Proof. In both cases, by Rouche's theorem $f(z) = w$ has a root for every $w \in \mathbb{D}$ if $f(z) = 0$ has a root. But if $f(z) = 0$ has no root, then $1/f$ defined on $\overline{\mathbb{D}}$ achieves its maximum in the interior $\mathbb{D}$ (by maximum principle for the first case, obvious in the second case). □

Exercise 3.I. [SSh03, 3.19] *Prove the maximum principle for harmonic functions.*

Proof. Suppose an harmonic function u defined on an open set Ω achieves a local maximum M at $z_0 \in \Omega$. We know that there exists a holomorphic function f on Ω such that $\mathrm{Re}(f) = u$. Then f is not open since $f(z_0) = M + ib$, and no neighborhood of $M + ib$ is contained in the image $f(D)$ where D is a small neighborhood of z_0. □

Exercise 3.J (Laurent Series Expansion)**.** [SSh03, Problem 3.3] *Suppose f is analytic on a region containing the annulus $\{r_1 \leq |z - z_0| \leq r_2\}$. Then, we can write (uniquely)*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the series converges absolutely in the interior of the annulus.

Proof. By Theorem 2.14, one can write

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

and use the series expansion of $1/(\zeta - z) = \frac{1}{(\zeta - z_0) - (z - z_0)}$ appropriately in each case. □

4. Conformal Maps

4.1. Conformal equivalence and examples.

Proposition 4.2. *If $f : U \to V$ for $U, V \subset \mathbb{C}$ open is holomorphic and injective, then $f'(z_0) \neq 0$ for all $z_0 \in U$. Moreover, as a result the inverse of f defined on its image is holomorphic.*

Proof. Write $f(z) - f(z_0) = a_k(z - z_0)^k + [(z - z_0)^{k+1}]$ and use Rouche's theorem to conclude that $f(z) - f(z_0)$ is not injective. Second part follows: $(f^{-1})'(f(z_0)) = \frac{1}{f'(z_0)}$. □

Definition 4.3. *A map holomorphic map $f : U \to V$ with $f'(z_0) \neq 0$ $\forall z_0 \in U$ is called* ***conformal map****. If f is bijective, then it is called a* ***biholomorphism*** *(note that its inverse is also holomorphic), in which we say U, V are* ***conformally equivalent****.*

Example 4.4. Translations $z \mapsto z+a$ and rotation+dilation given by $z \mapsto cz$, $(c \in \mathbb{C})$ are conformal equivalences $\mathbb{C} \xrightarrow{\sim} \mathbb{C}$.

Example 4.5. Let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ be the upper half-plane. $\mathbb{H}$ and the unit disk $\mathbb{D}$ are conformally equivalent. One equivalence is given by $F : \mathbb{H} \to \mathbb{D}, G : \mathbb{D} \to \mathbb{H}$ where

$$F(z) = \frac{i - z}{i + z}, \quad G(w) = i\frac{1 - w}{1 + w}$$

Example 4.6. For $0 < \alpha < 2$, the map $f(z) = z^\alpha$ defined in terms of the principal branch is a biholomorphic map from $\mathbb{H}$ to the sector $S = \{w \in \mathbb{C} : \ 0 < \arg(w) < \alpha\pi\}$.

Example 4.7. The map $f(z) = \log z$ is a biholomorphism from $\mathbb{H}$ to a region $\{a + bi : a \in \mathbb{R}, 0 < b < \pi\}$. It also biholomorphically maps upper unit disk to $\{a + bi : a < 0, 0 < b < \pi\}$

4.8. The Mobius transformations.

Definition 4.9. *We call maps of the following form a* ***Mobius transformation / (fractional) linear map****:*

$$f(z) = \frac{az + b}{cz + d}$$

for $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Remark 4.10. Noting the identification $\mathbb{CP}^1 \simeq \widehat{\mathbb{C}}$, we see that a Mobius map computed in $\mathbb{CP}^1$ is $[z_1 : z_2] \to [az_1 + bz_2 : cz_1 + dz_2]$. In other words, it really is a linear transformation in homogeneous coordinates made by multiplying matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. In this view, one sees that matrices of $PSL_2(\mathbb{C})$ correspond exactly to different Mobius maps, and so a Mobius map is determined by image of three distinct points. Moreover, composition of Mobius maps corresponds to matrix multiplication. Indeed, it is thus a biholomorphic map $\widehat{\mathbb{C}} \xrightarrow{\sim} \widehat{\mathbb{C}}$. Moreover,

Proposition 4.11. *Given three distinct points $z_2, z_3, z_4 \in \widehat{\mathbb{C}}$, the Mobius map T that maps z_2, z_3, z_4 to $1, 0, \infty$, respectively, is given by*

$$f(z) = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}$$

(if $z_2, z_3,$ or $z_4 = \infty$, just cancel the terms with it). We denote the above $f(z)$ by (z, z_2, z_3, z_4) called the ***cross ratio****.*

Theorem 4.12. *For distinct points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ and T a Mobius map, $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$. And hence, T that maps $z_2, z_3, z_4)$ to w_2, w_3, w_3 is obtained by writing $(w, w_2, w_3, w_4) = (z, z_2, z_3, z_4)$ and solving for w.*

Example 4.13. Fractional linear map gives us abundance of biholomorphism, especially when we use them to rotate the Riemann sphere. The map $z \mapsto (z, i, 1, -1) = \frac{(z-1)(i+1)}{(z+1)(i-1)} = i\frac{1-z}{1+z}$ is the map $G : \mathbb{D} \to \mathbb{H}$ in Example 4.5. In another case, $z \mapsto (z, 0, -1, 1) = \frac{(z+1)(-1)}{(z-1)(1)} = \frac{1+z}{z-1}$ maps upper half-disk to the first quadrant.

4.14. **The Schwarz lemma and** $\mathrm{Aut}(\mathbb{D})$, $\mathrm{Aut}(\mathbb{H})$**.**

Proposition 4.15 (Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$, and if equality occurs at $z_0 \in \mathbb{D}$, then f is a rotation. Moreover, $|f'(0)| \leq 1$, and if equal then f is a rotation.*

Proof. Consider the holomorphic function $\frac{f(z)}{z}$ and use the maximum principle. □

Definition 4.16. *For a open set $\Omega \subset \mathbb{C}$, an **automorphism** of Ω is a biholomorphic map $f : \Omega \to \Omega$. Automorphisms of Ω forms a group $\mathrm{Aut}(\Omega)$.*

Example 4.17. In [SSh03, Exercise 3.14], we proved that $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C},\ a \neq 0\}$.

Theorem 4.18. *Automorphisms of $\mathbb{D}$ are exactly the maps*

$$f(z) = e^{i\theta}\frac{\alpha - z}{1 - \overline{\alpha}z}$$

where $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$.

Proof. Note that the map $\varphi_\alpha(z) := \frac{\alpha - z}{1-\overline{\alpha}z}$ is a biholomorphism $\mathbb{D} \to \mathbb{D}$ that exchanges 0 and α, and φ_α is its own inverse. Now, suppose $f \in \mathrm{Aut}(\mathbb{D})$ and $f(0) = \alpha$. Consider $g = f \circ \varphi_\alpha$, which biholomorphically maps $\mathbb{D} \to \mathbb{D}$ and $g(0) = 0$. By Schwarz lemma on both g and g^{-1}, we get $|g(z)| = |z|$ for $z \in \mathbb{D}$, and hence g is a rotation $g = e^{i\theta}$. But then $f = g \circ \varphi_\alpha$. □

Corollary 4.19. *Automorphisms of $\mathbb{D}$ that fix the origin are the rotations.*

Theorem 4.20. *Automorphisms of $\mathbb{H}$ are exactly of the form*

$$z \mapsto \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$. In other words, we have an isomorphism

$$\mathrm{Aut}(\mathbb{H}) \simeq PSL_2(\mathbb{R})$$

Proof. Let $F : \mathbb{H} \to \mathbb{D}$ be a biholomorphism. Note the isomorphism $\mathrm{Aut}(\mathbb{D}) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{H})$ via $f \mapsto F^{-1} \circ f \circ F$. Then, the previous theorem and computation yields the desired result. □

Remark 4.21. Note that $\mathrm{Aut}(\mathbb{D}), \mathrm{Aut}(\mathbb{H})$ act transitively on $\mathbb{D}, \mathbb{H}$ (respectively), but not faithfully.

4.22. **The Riemann mapping theorem.**

Before stating and proving the Riemann mapping theorem and its proof, we consider some metric topological matters.

Given a metric space (X, d), X is **totally bounded** if X can be covered by finitely many ϵ-balls for any given $\epsilon > 0$. It is well-known that

A metric space X is compact iff it is complete and totally bounded

Given a metric space (Y, d) and X a set, we can define a metric on Y^X by

$$\rho(f, g) := \begin{cases} \sup_{x \in X} d(f(x), g(x)) \\ 1 \text{ if } \sup > 1 \end{cases}$$

This is the **uniform topology** on Y^X; convergence in this metric is exactly uniform convergence of functions. Hence, we know that $C(X, Y) \subset Y^X$ is closed. Moreover, note the fact that Y^X is complete if Y is complete. Note that unit-ball in $C(X,Y)$ is not compact; e.g. $\{x^n\}_n \subset C([0,1])$ is not sequentially compact. For K a compact metric space, a family of functions $\mathscr{F} \in C(K)$ is **uniformly bounded** if there exists M such that $|f| \leq M \ \forall f \in \mathscr{F}$, and $\mathscr{F}$ is **equicontinuous** if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in K, \ d(x,y) < \delta$ and $f \in \mathscr{F}$.

Theorem 4.23 (Arzela-Ascoli)**.** *Let K is a compact metric space. If a family of functions $\mathscr{F} \subset C(K)$ is equicontinuous and uniformly bounded, then $\overline{\mathscr{F}}$ is compact.*

In complex analysis, a related notion to a family of functions being compact is the following:

Definition 4.24. *Let $\Omega \subset \mathbb{C}$ be open, and $\mathscr{F}$ be a family of holomorphic functions on Ω. $\mathscr{F}$ is* ***normal*** *if every sequence in $\mathscr{F}$ has a subsequence that converges uniformly on every compact subset of Ω (limit need not be in $\mathscr{F}$).*

Theorem 4.25 (Montel's theorem)**.** *Let $\mathscr{F}$ be a family of holomorphic functions on Ω. If $\mathscr{F}$ is uniformly bounded on every compact subset of Ω, then $\mathscr{F}$ is equicontinuous on every compact subset of Ω, and hence $\mathscr{F}$ is a normal family.*

Proof. Note that if $|f'|$ is bounded, then f is Lipschitz cotninuous, so use Cauchy integral formula and that $\mathscr{F}$ is uniformly bounded to show that $|f'(z)| \leq M$ for all $f \in \mathscr{F}$ and $z \in \Omega$. This show $\mathscr{F}$ equicontinuous. Then use Arzela-Ascoli theorem with exhaustion of Ω by compact sets to show normal. ☐

Proposition 4.26. *If Ω is a region and $\{f_n\}$ a sequence of injective holomorphic functions on Ω that converges uniformly to a holomorphic function f on every compact subset of Ω, then f is either injective or constant.*

Proof. If $f(z_1) = f(z_2)$, then consider the sequence $g_n(z) := f_n(z) - f(z_1)$. Note that $g_n \to g := f(z) - f(z_1)$ uniformly on all compact subsets and so does $g_n' \to g'$. Thus, for a small circle around z_2, we must have $0 = \frac{1}{2\pi i}\int_C \frac{g_n'(z)}{g_n(z)}dz = \frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)}dz = 1$, which is a contradiction. ☐

Theorem 4.27 (Riemann mapping theorem)**.** *If $\Omega \subset \mathbb{C}$ is proper and simply-connected region, then for $z_0 \in \Omega$, there exists a unique biholomorphism $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.*

Proof. TODO ☐

4.28. **Exercises.**

Exercise 4.A. [SSh03, 8.10] *Let $F : \mathbb{H} \to \mathbb{C}$ be a holomorphic function satisfying $|F(z)| \leq 1$ and $F(i) = 0$. Then show that $|F(z)| \leq |\frac{i-z}{i+z}|$.*

Proof. Note that $G : \mathbb{H} \to \mathbb{D}$ defined by $G(z) := \frac{i-z}{i+z}$ and $G^{-1}(w) = i\frac{1-w}{1+w}$ is a conformal equivalence. Define $H : \mathbb{D} \to \mathbb{D}$ by $H := F \circ G^{-1} : \mathbb{D} \to \mathbb{C}$. Since H maps $\mathbb{D} \to \mathbb{D}$ and $H(0) = 0$, by the Schwarz lemma we have $|H(w)| \leq |w|$ for all $w \in \mathbb{D}$. In other words, $|F(G^{-1}(w))| \leq |G(G^{-1}(w)|$, and thus $|F(z)| \leq |z|$ for $z \in \mathbb{H}$. ☐

Exercise 4.B. [SSh03, 8.12] *If $f : \mathbb{D} \to \mathbb{D}$ is analytic and has two distinct fixed points, then f is the identity (i.e. $f(z) = z$).*

Proof. Suppose $\alpha, \beta \in \mathbb{D}$ are two distinct fixed points. Consider the biholomorphism $\phi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} z}$, which satisfies $\phi_\alpha(0) = \alpha$ and $\phi_\alpha(\beta) = \beta'$ (note $\phi_\alpha(\beta') = \beta$). Now, consider the map $g := \phi_\alpha \circ f \circ \phi_\alpha : \mathbb{D} \to \mathbb{D}$, which has fixed points 0 and β'. By the Schwarz lemma, g is a rotation that fixes a nonzero point, and hence identity, and thus f is also identity. ☐

Exercise 4.C. [SSh03, 8.14] *Show that all biholomorphic maps $\mathbb{H} \to \mathbb{D}$ take the form*

$$z \mapsto e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}, \quad \theta \in \mathbb{R},\ \beta \in \mathbb{H}$$

Proof. Any biholomorphism $f : \mathbb{H} \to \mathbb{D}$ factors through as $f = (f \circ F^{-1}) \circ F$ where $F : \mathbb{H} \to \mathbb{D}$ is a biholomorphism $z \mapsto \frac{i-z}{i+z}$ and $f \circ F^{-1} \in \mathrm{Aut}(\mathbb{D})$ is of the form $z \mapsto e^{i\theta}\frac{\alpha-z}{1-\overline{\alpha}z}$ for $\theta \in \mathbb{R},\ \alpha \in \mathbb{D}$. Now, computing the composition of the Mobius transformation

$$e^{i\theta}\begin{bmatrix}-1 & \alpha\\ -\overline{\alpha} & 1\end{bmatrix}\begin{bmatrix}-1 & i\\ 1 & i\end{bmatrix} = e^{i\theta}\begin{bmatrix}\alpha+1 & i(\alpha-1)\\ \overline{\alpha}+1 & i(1-\overline{\alpha})\end{bmatrix}$$

which factors as

$$e^{i\theta}\begin{bmatrix}\alpha+1 & 0\\ 0 & \overline{\alpha}+1\end{bmatrix}\begin{bmatrix}1 & -\beta\\ 1 & -\overline{\beta}\end{bmatrix}$$

where $\beta = i\frac{1-\alpha}{1+\alpha} = F^{-1}(\alpha)$. Since $|\alpha+1| = |\overline{\alpha}+1|$, the left matrix also rotation Mobius map. Hence, for some θ' and $\beta \in \mathbb{H}$ as defined, $f(z) = e^{i\theta'}\frac{z-\beta}{z-\overline{\beta}}$, as desired. □

Exercise 4.D. [SSh03, 8.15] *Suppose $\Phi \in \mathrm{Aut}(\mathbb{H})$ that fixes three distinct points on the real axis, then Φ is identity. If (x, y, z) and (x', y', z') are two pairs of three distinct points on the real axis with $z_1 < z_2 < z_3,\ w_1 < w_2 < w_3$, then there exists a unique automorphism $\Phi \in \mathrm{Aut}(\mathbb{H})$ such that $\Phi(x_i) = w_i$. Same holds if $w_2 < w_3 < w_1$ or $w_3 < w_1 < w_2$.*

Proof. $\mathrm{Aut}(\mathbb{H}) \subset \mathrm{Aut}(\widehat{\mathbb{C}})$ as $PSL_2(\mathbb{R}) \subset PSL_2(\mathbb{C})$. Thus, since a Mobius transformation is determined by images of three distinct points, the first statement follows. Now, for the second statement, writing $(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3)$ and solving for w gives a Mobius transformation $\frac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{R}$ mapping z_1, z_2, z_3 to w_1, w_2, w_3, and with (a lot of) computation, one checks that $ad - bc > 0$ (so that $\begin{bmatrix}a & b\\ c & d\end{bmatrix} \in PSL_2(\mathbb{R})$) exactly when w_i's are ordered as given. □

Exercise 4.E. [SSh03, Problem 8.2] *The* ***oriented angle*** *of $z, w \in \mathbb{C}$ is determined by two quantities*

$$\frac{\langle z, w\rangle}{|z||w|} \text{ and } \frac{\langle z, -iw\rangle}{|z||w|}, \quad \text{where } \langle z, w\rangle = \mathrm{Re}(w\overline{z})$$

An oriented angle of two intersecting curves at the intersection is defined as the angle of two tangent vectors at the intersection. A map $f : \Omega \to \mathbb{C}$ is ***angle-preserving at*** *$z_0 \in \Omega$ if for any two curves $\gamma, \eta \subset \Omega$ intersecting at z_0, the (oriented) angle of γ, η at z_0 and the angle of $f \circ \gamma, f \circ \eta$ at $f(z_0)$ are the same. Show that:*

1. *If $f : \Omega \to \mathbb{C}$ is holomorphic with $f(z_0) \neq 0$, then f is angle-preserving at z_0.*
2. *Conversely, if $f : \Omega \to \mathbb{C}$ is real-differentiable at z_0 with $J_f(z_0) \neq 0$ and is angle-preserving, then f is holomorphic at z_0.*

Proof. (1) is easy, for if $\gamma(t_0) = z_0$ and $\eta(s_0) = z_0$, then $(f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0),\ (f \circ \eta)'(t_0) = f'(z_0)\eta'(t_0)$. For the converse, by chain rule, if γ is a curve through z_0 at t_0, then $[Df]_{z_0}\gamma'(t_0) = (f \circ \gamma)'(t_0)$. Since the matrix $M := [Df]_{z_0}$ is such that $\langle u, v\rangle = \langle Mu, Mv\rangle$ and $\langle u, -iv\rangle = \langle Mu, M(-iv)\rangle$ for any $|u| = |v| = 1$, it is of the form $\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$, which means that f satisfies the Cauchy-Riemann equation at z_0. □

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