Combinatorial Hodge theory

Christopher Eur [*] Matt Larson [†]

October 31, 2024

Developed classically in the context of complex geometry, Hodge theory gives certain rigid structures on the cohomology rings of compact Kähler manifolds. In the past few decades, it has been discovered that these structures exist in other contexts as well. Finding such "Hodge theoretic" structures associated to combinatorial objects has led to remarkable developments, including resolutions of long-standing open problems in combinatorics. Here, we give a broad (and necessarily incomplete) snapshot of these developments in combinatorial Hodge theory. Previous surveys on this topic, with more extensive lists of references, can be found at [Huh23, Eur24].

In Section 1, we define the structure colloquially known as the "Kähler package" and discuss some general principles for its combinatorial applications. These general principles are illustrated in concrete examples in Sections 2, 3, and 4. These sections each feature a different combinatorial object, but they follow a common template outlined in Section 1.3, so the readers may pick and choose to their taste. In Section 5, we outline some strategies common to the proofs of the "Kähler package" for many different combinatorial structures.

1 The Kähler package

Let us begin with a toy example.

Example 1. Let $A^\bullet$ be the polynomial ring $\mathbb{R}[x, y, z]$ modulo the ideal $\langle x^2, y^2, z^2\rangle$. For each i, the degree i graded component A^i, as a $\mathbb{R}$-vector space, has a basis (as an $\mathbb{R}$-vector space) given by the square-free monomials:

$$\begin{aligned} A^0 &= \text{span}_{\mathbb{R}}(1)\\ A^1 &= \text{span}_{\mathbb{R}}(x, y, z)\\ A^2 &= \text{span}_{\mathbb{R}}(xy, yz, xz)\\ A^3 &= \text{span}_{\mathbb{R}}(xyz). \end{aligned}$$

While the symmetry in the dimensions of the graded components is apparent, there are more refined structures. For example, multiplication by $(x + y + z)$ is an isomorphism $A^1 \to A^2$, and multiplication by $(x + y + z)^3$ is an isomorphism $A^0 \to A^3$.

One may recognize the ring in the toy example as the cohomology ring of $(\mathbb{CP}^1)^3$. The example gives a glimpse of the following property, which was first discovered in the cohomology rings of compact Kähler manifolds.[1]

Definition 2. Let $A^\bullet$ be a graded $\mathbb{R}$-algebra $\oplus_{i=0}^d A^i$ which is finite dimensional as a $\mathbb{R}$-vector space, together with an isomorphism $\deg\colon A^d \to \mathbb{R}$ and a nonempty open convex cone $\mathcal{K} \subset A^1$. We say that $(A^\bullet, \deg, \mathcal{K})$ *satisfies the Kähler package* if:

(PD) For each $0 \le i \le d$, the symmetric bilinear form $A^i \times A^{d-i} \to \mathbb{R}$ given by $(a, b) \mapsto \deg(a \cdot b)$ is nondegenerate.

(HL) For each $0 \le i \le d/2$ and any $\ell_1, \dots, \ell_{d-2i} \in \mathcal{K}$, the map $A^i \to A^{d-i}$ given by multiplication by $\ell_1 \cdots \ell_{d-2i}$ is an isomorphism.

[*] Carnegie Mellon University. Email: ceur@cmu.edu.
[†] Princeton University. Email: mattlarson@princeton.edu.

[1] More precisely, for a projective Kähler manifold X, the subring $\bigoplus_p H^{p,p}(X)$ of real (p, p)-forms in its cohomology ring satisfies the property stated as the "Kähler package" here. See [DN06] for a proof.

(HR) For each $0 \leq i \leq d/2$ and any $\ell_0, \ldots, \ell_{d-2i} \in \mathcal{K}$, the bilinear form $A^i \times A^i \to \mathbb{R}$ given by $(a, b) \mapsto (-1)^i \deg(a \cdot \ell_1 \cdots \ell_{d-2i} \cdot b)$ is positive definite on the kernel of multiplication by $\ell_0 \ell_1 \cdots \ell_{d-2i}$.

The reader may verify these properties for the ring in the toy example, where the cone $\mathcal{K}$ is $\{ax+by+cz : a, b, c > 0\}$ and the isomorphism $\deg\colon A^3 \to \mathbb{R}$ is given by $\deg(xyz) = 1$.

(PD), (HL), and (HR) stands for "Poincaré duality property," "hard Lefschetz property," and "Hodge–Riemann relations," after those who discovered such properties in topology and complex geometry. See Figure 1 for a visualization of the Kähler package.

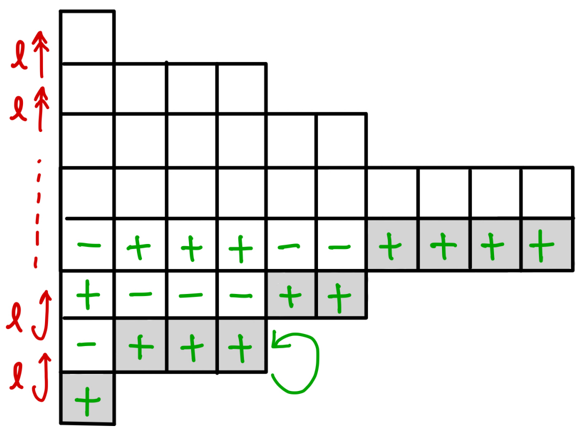


Figure 1: The rows represent the graded components $A^0, A^1, \ldots, A^d$, with the number of boxes equal to the dimension. The gray boxes represent the kernel of multiplication by $\ell_0 \ell_1 \cdots \ell_{d-2i}\colon A^i \to A^{d-i+1}$. (PD) implies that the diagram is symmetric; (HL) implies that the diagram monotonically widens towards the middle; (HR) and (HL) imply that the signature of the bilinear form $A^i \times A^i \to \mathbb{R}$ is as illustrated.

Remark 3. One sometimes works with the following more general setup for the Kähler package, at the cost of losing the ring structure. That is, instead of a graded $\mathbb{R}$-*algebra*, let $A^\bullet$ be a (finite-dimensional) graded $\mathbb{R}$-vector space $\bigoplus_{i=0}^d A^i$, together with a symmetric bilinear form $\deg\colon A^\bullet \times A^{d-\bullet} \to \mathbb{R}$ and an open convex subset $\mathcal{K}$ of commuting linear operators $L\colon A^\bullet \to A^{\bullet+1}$ that satisfy $\deg(L(\cdot), \cdot) = \deg(\cdot, L(\cdot))$. Then (PD), (HL), and (HR) can be formulated as before. Geometrically, this more general setup arises when one extends beyond manifolds to study spaces with singularities, where a "finer" invariant than singular cohomology, known as *intersection cohomology*, is useful. Intersection cohomology in general does not have a ring structure.

Suppose now that for a combinatorial object X of "dimension" d, one can construct a graded $\mathbb{R}$-algebra $A^\bullet(X) = \bigoplus_{i=0}^d A^i(X)$ that encodes certain combinatorial data about X. Examples of such X and $A^\bullet(X)$ will be illustrated in Sections 2, 3, and 4. The validity of the Kähler package for $A^\bullet(X)$ can then give highly nontrivial information about X. We feature two such ways in the next two subsections. Throughout, suppose that $A^\bullet = A^\bullet(X)$ satisfies the Kähler package.

1.1 The hard Lefschetz property and Hilbert functions

Let us consider the Hilbert function, that is, the sequence $(a_0, \ldots, a_d)$ of the dimensions $a_i = \dim A^i$ of the graded pieces of $A^\bullet$. Firstly, Poincaré duality (PD) implies the symmetry

$$a_i = a_{d-i} \quad \text{for each } i \in \{0, \ldots, d\},$$

but the hard Lefschetz property (HL) implies an even more rigid restriction on the Hilbert function: for any $\ell \in \mathcal{K}$ and $i \leq d/2$, the multiplication map $\ell\colon A^i \to A^{i+1}$ must be injective for the multiplication map $\ell^{d-2i} : A^i \to A^{d-i}$ to be an isomorphism. Therefore, (HL) implies

$$1 \leq a_1 \leq \cdots \leq a_{\lfloor d/2 \rfloor},$$

or more strongly, that the sequence of consecutive differences $(1, a_1 - 1, a_2 - a_1, \ldots, a_{\lfloor d/2 \rfloor} - a_{\lfloor d/2 \rfloor - 1})$ is the Hilbert function of the quotient $A^\bullet/\langle \ell \rangle$.

If furthermore $A^\bullet$ is generated as an algebra in degree 1, then $A^\bullet/\langle \ell \rangle$ is also. Macaulay classified the Hilbert functions of graded algebras which are generated in degree 1, as follows. Given positive integers b and i, one can uniquely write b as

$$b = \binom{k_i}{i} + \binom{k_{i-1}}{i-1} + \cdots + \binom{k_j}{j} \text{ for } k_i > \cdots > k_j \geq 1.$$

Defining $b^{\langle i\rangle} = \binom{k_i+1}{i+1} + \cdots + \binom{k_j+1}{j+1}$, we say that a sequence $(1, b_1, \ldots, b_m)$ is a *Macaulay vector* (or an *O-sequence*) if $0 \le b_{i+1} \le b_i^{\langle i\rangle}$ for all $i \ge 1$. Macaulay showed that $(1, b_1, \ldots, b_m)$ is the Hilbert function of a finite dimensional graded algebra which is generated in degree 1 if and only if it is a Macaulay vector [HMM+13, Section 6.2]. We deduce that

$$(1, a_1 - 1, a_2 - a_1, \ldots, a_{\lfloor d/2\rfloor} - a_{\lfloor d/2\rfloor-1}) \quad \text{is a Macaulay vector.}$$

The properties here only require the existence of one element $\ell \in A^1$ satisfying the condition (HL). For a treatment of commutative rings with such an element, see [HMM+13].

1.2 Hodge–Riemann relations and log-concavity

Another application of the Kähler package concerns intersection numbers, i.e., numbers obtained by applying the map $\deg\colon A^d \to \mathbb{R}$ to certain products of elements in $A^\bullet$. In particular, for any choice of α and β in A^1, one has a sequence $(m_0, m_1, \ldots, m_d)$ defined by $m_j = \deg(\alpha^j\beta^{d-j})$. The Hodge–Riemann relations (HR) with $i = 0$ and $i = 1$ imply the following properties of this sequence.

Proposition 4. When α and β lie in $\overline{\mathcal{K}}$, the closure of $\mathcal{K}$ in the real vector space A^1, the sequence $(m_0, \ldots, m_d)$ defined above

- is *non-negative*, i.e., $m_j \ge 0$ for all $0 \le j \le d$,
- is *log-concave*, i.e., $m_j^2 \ge m_{j-1}m_{j+1}$ for all $1 \le j \le d-1$, and
- has *no internal zeros*, i.e., if $m_j \ne 0$ and $m_k \ne 0$ for some $j < k$ then $m_l \ne 0$ for all $j < l < k$.

In particular, it is *unimodal*, i.e., $m_0 \le \cdots \le m_k \ge m_{k+1} \ge \cdots \ge m_d$ for some $0 \le k \le d$.

Proof. The first two claimed properties are closed conditions, so let us first consider the case where $\alpha, \beta \in \mathcal{K}$. Then, (HR) with $i = 0$ implies that $m_j = \deg(1 \cdot \alpha^j\beta^{d-j} \cdot 1) > 0$ for all $0 \le j \le d$.

For the log-concavity, if α and β are linearly dependent, we have $m_j^2 = m_{j-1}m_{j+1}$ for all j, so assume now that they are linearly independent. For a fixed j, define the bilinear form $Q\colon A^1 \times A^1 \to \mathbb{R}$ by $(a, b) \mapsto \deg(a \cdot \alpha^{j-1}\beta^{d-j-1} \cdot b)$. Then, (HR) with $i = 1$ states that Q is negative definite on a codimension-1 subspace of A^1, so Q cannot restrict to be positive definite on $\operatorname{span}_{\mathbb{R}}(\alpha, \beta)$. Since $m_d = Q(\alpha, \alpha) > 0$, the restriction of Q to the 2-dimensional subspace $\operatorname{span}_{\mathbb{R}}(\alpha, \beta)$ is not negative definite. Hence, we conclude that $\det \begin{bmatrix} Q(\alpha,\alpha) & Q(\alpha,\beta) \\ Q(\alpha,\beta) & Q(\beta,\beta) \end{bmatrix} \le 0$, or, equivalently, that $m_{j-1}m_{j+1} - m_j^2 \le 0$, as desired.

Lastly, a limit of positive and log-concave sequences with no internal zeros is non-negative and log-concave with no internal zeros, yielding the desired result when $\alpha, \beta \in \overline{\mathcal{K}}$. □

In degrees higher than 1, it is difficult to deduce general inequalities from the Hodge–Riemann relations in a similar way as we have done for log-concavity here. However, the Hodge–Riemann relations in higher degrees are often useful for proving (HL) and provide additional restrictions on the structure of $A^\bullet$.

1.3 A template

The next three sections will each match the following common template, consisting of five parts.

(1) Objects (What is X?): We state the combinatorial objects X of interest. While we assume a passing familiarity with them, explicit examples are provided for the uninitiated.

(2) Questions (What about X?): We discuss questions about X that were resolved by the development of combinatorial Hodge theory for X.

(3) Algebras (What is $A^\bullet(X)$?): We define the algebra $A^\bullet(X)$ that satisfies the Kähler package and explain how it was used to resolve questions about X.

(4) Geometric origin (Where is $A^\bullet(X)$ from?): We explain how, for a certain subset of the objects X, the ring $A^\bullet(X)$ has a geometric origin as the cohomology ring of a complex projective manifold.

(5) "Singular" objects (How about beyond X?): Often, there is a natural enlargement of the class of combinatorial objects X. Or, a different question about X leads to an algebra different from the one featured in part (3). In both cases, a naive candidate for $A^\bullet(X)$ often fails the Kähler package, but we discuss briefly how one can develop a combinatorial "intersection cohomology" module to amend this failure.[2]

2 Polytopes

(1) Objects.

Definition 5. A *polytope* P in $\mathbb{R}^d$ is a bounded intersection of finitely many closed half-spaces. A subset of P is a *face* of P if it is the locus where a linear functional on $\mathbb{R}^d$ achieves its minimum on P, and it is a *facet* if it is a maximal proper face. We say P is *simplicial* if every proper face of P is a simplex.

We assume that P is full-dimensional in $\mathbb{R}^d$ and primarily consider simplicial polytopes. We point to [Zie95] as a general reference on polytopes.

Figure 2: Two polytopes in $\mathbb{R}^3$. The bipyramid (two regular tetrahedra glued along a triangle) on the left is simplicial, but the cube on the right is not.

(2) Questions. Of enduring interest in combinatorics is the structure of the collection of faces of a polytope P. A basic starting point is the number of faces of each dimension. For $-1 \le i \le d-1$, let

$$f_i = |\{i\text{-dimensional faces of } P\}|$$

with $f_{-1} = 1$ for the empty face. The sequence $(f_{-1}, f_0, \ldots, f_{d-1})$ is called the *f-vector*. For example, in Figure 2 the bipyramid has f-vector $(1, 5, 9, 6)$ and the cube has f-vector $(1, 8, 12, 6)$.

An intriguing structure is revealed when one applies the following invertible change of coordinates to the f-vector of a polytope. Define the *h-vector* $(h_0, \ldots, h_d)$ of P by

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

Equivalently, the h-vector is defined by the relation

$$f(t-1) = h(t)$$

where $f(t)$ and $h(t)$ are the polynomials

$$f(t) = f_{-1}t^d + f_0 t^{d-1} + \cdots + f_{d-1} \text{ and}$$
$$h(t) = h_0 t^d + h_1 t^{d-1} + \cdots + h_d.$$

For example, the bipyramid has h-vector $(1, 2, 2, 1)$ and the cube has h-vector $(1, 5, -1, 1)$. Let us make two observations from these two examples:

- We have $h_d = 1$ for both. Indeed, $h_d = 1$ holds in general because $h_d = (-1)^{d-1} \sum_{i=0}^{d} (-1)^{i-1} f_{i-1}$ is the alternating sum of the face numbers, and the reduced Euler characteristic of the boundary of P (which is a $(d-1)$-dimensional sphere) is $(-1)^{d-1}$.
- For the bipyramid, a simplicial polytope, the h-vector is positive, symmetric, and unimodal.

Do these properties persist for all simplicial polytopes? That is, are h-vectors of simplicial polytopes always positive, symmetric, and unimodal? Similar questions about non-simplicial polytopes will be discussed in part (5). For now, we consider the following conjecture of McMullen:

*A sequence $(h_0, \ldots, h_d)$ of integers is the h-vector of a simplicial polytope if and only if it is symmetric and the sequence $(h_0, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$, called the g-*vector *of P, is a Macaulay vector.*

Note that the condition about the g-vector implies the positivity and unimodality of $(h_0, \ldots, h_d)$ since

[2] While this is often a fascinating and intricate part of the story about the combinatorial object X, our discussion is kept short due to the introductory nature of this survey.

$h_0 = f_{-1} = 1$. Billera and Lee [BL80] showed by explicit construction that if a sequence $(h_0, \ldots, h_d)$ satisfies the stated conditions, then there exists a simplicial polytope with the sequence as its h-vector. For the converse, i.e., that every h-vector of a simplicial polytope satisfies these conditions, the proof uses the Kähler package of the following algebra associated with a simplical polytope P.

(3) Algebras. Without loss of generality, translate P so that it contains the origin in its interior.

Definition 6. Let $\mathbb{R}[x_v : v \text{ a vertex of } P]$ be the polynomial ring whose variables are labelled by the vertices of P. Let $A^\bullet(P)$ be the quotient ring

$$A^\bullet(P) = \frac{\mathbb{R}[x_v]}{I + J}$$

where I and J are ideals defined by

$I = \langle x_{v_1} \cdots x_{v_k} : \{v_1, \ldots, v_k\} \text{ not a face of } P\rangle$ and
$J = \langle \sum_v f(v) x_v : f \text{ a linear function on } \mathbb{R}^d\rangle$.

Note that $A^\bullet(P)$ is graded. A key property of $A^\bullet(P)$ is that the dimensions of the graded pieces are given by the h-vector of P [Sta75]. For example, placing the origin at the center of the base triangle of the bipyramid in Figure 2, one finds that in this case

$$A^\bullet(P) = \frac{\mathbb{R}[x_1, x_2, x_3, x_4, x_5]}{\langle x_1x_2x_3, x_4x_5\rangle + \langle x_1 - x_2, x_2 - x_3, x_4 - x_5\rangle},$$

where the variables x_1, x_2, and x_3 correspond to the vertices of the central simplex, and the variables x_4 and x_5 correspond to the top and bottom vertices. We see that this ring is isomorphic to $\mathbb{R}[x_1, x_4]/\langle x_1^3, x_4^2\rangle$, whose graded components A^0, A^1, A^2, A^3 have bases $\{1\}$, $\{x_1, x_4\}$, $\{x_1^2, x_1x_4\}$, $\{x_1^2x_4\}$, respectively.

Let us further make two observations about $A^\bullet(P)$ in general:

(i) As $A^d(P)$ is 1-dimensional (since $h_d = 1$), we may choose an isomorphism $\deg\colon A^d(P) \xrightarrow{\sim} \mathbb{R}$. There is such a choice such that, for every maximal proper face $F = \{v_1, \ldots, v_d\}$ of P, we have $\deg(x_{v_1} \cdots x_{v_d}) > 0$.

(ii) Let us say that a function φ is *piecewise linear* if, for every proper face F of P, the function φ is linear on the cone $\mathbb{R}_{\geq 0}\{v_i : i \in F\}$. The degree 1 component $A^1(P)$ can be identified with the space of *piecewise linear functions* modulo the globally linear functions on $\mathbb{R}^d$, as follows. Because P is simplicial, each element $D = \sum_v c_v x_v$ in the degree 1 part of $\mathbb{R}[x_v : v \text{ a vertex}]$ uniquely defines a piecewise linear function φ_D by the assignment $\varphi_D(v) = c_v$ for every vertex v. Under this correspondence, a linear function f on $\mathbb{R}^d$ is associated with $\sum_v f(v)x_v$, so the generators of the ideal J correspond to globally linear functions.

We say that $\ell \in A^1(P)$ is *ample* if, for every proper face F of P, we can choose a piecewise linear function representing ℓ which vanishes on F and is strictly positive on all vertices not contained in F.[3] The ample elements of $A^1(P)$ form an open cone $\mathcal{K}$. This cone is nonempty: for instance, consider the piecewise linear function which takes value 1 on each maximal proper face of P.

Theorem 7 ([Sta80a, McM93]). Let P be a simplicial polytope. The triple $(A^\bullet(P), \deg, \mathcal{K})$ satisfies the Kähler package.

Because $A^\bullet(P)$ is a graded algebra generated in degree 1, from our discussion in Section 1.1 about applications of the hard Lefschetz property, we conclude that the h-vector is symmetric and the g-vector is a Macaulay vector, i.e., McMullen's conjecture describing the possible h-vectors of simplicial polytopes holds.

(4) Geometric origin. Stanley proved that $A^\bullet(P)$ has the Kähler package by observing that it is the cohomology ring of a projective *toric variety*. The construction of this toric variety involves perturbing the coordinates of the vertices of P so that they have rational coordinates; this does not change the structure of the faces of P because P is simplicial. This toric variety is in general singular, but the singularities are mild enough that the real (p, p)-forms in its cohomology ring still have the Kähler package. McMullen

[3] This is equivalent to ℓ being a *strictly convex* piecewise linear function.

later gave a purely combinatorial proof that avoids perturbing the coordinates or constructing toric varieties.

(5) "Singular" objects. For non-simplicial polytopes, the h-vector no longer has the pleasant properties of the h-vector of simplicial polytopes. For example, we saw that the h-vector $(1, 5, -1, 1)$ of the cube is no longer non-negative. Instead, one should consider the *toric h-vector* [Sta87], which is a combinatorial invariant of a polytope P which is recursively defined in terms of the poset of faces of P. When P is simplicial, it agrees with the h-vector. The toric h-vector of the 3-dimensional cube is $(1, 5, 5, 1)$.

When the polytope has vertices with rational coordinates, the toric h-vector records the dimensions of the graded pieces of the intersection cohomology of the associated projective toric variety. The hard Lefschetz theorem for intersection cohomology implies that the toric h-vector is symmetric and unimodal.

For arbitrary polytopes, it may not be possible to perturb the vertices so that they have rational coordinates without changing the structure of the poset of faces, so there may be no associated toric variety. Nevertheless, Karu [Kar04] showed that the toric h-vector is symmetric and unimodal. He proved this by verifying that a version of the Kähler package (as formulated in Remark 3) holds for an analogue of the intersection cohomology of a projective toric variety.

3 Bruhat posets

(1) Objects. Let V be a finite dimensional real vector space with an inner product. For a hyperplane $H \subset V$ through the origin, the reflection across H defines an automorphism s_H of V.

Definition 8. A *finite Coxeter group* W (represented on V) is the subgroup of $GL(V)$ generated by the set of reflections $\{s_H\}_{H\in\mathcal{H}}$ where $\mathcal{H}$ is a finite set of hyperplanes in V satisfying $s_H(\mathcal{H}) = \mathcal{H}$ for all $H \in \mathcal{H}$.

We point to [Hum90] as a general reference on Coxeter groups.

Example 9. With the standard inner product on $\mathbb{R}^n$, the symmetric group $\mathfrak{S}_n$ of $\{1, 2, \dots, n\}$ is a finite Coxeter group represented on $\mathbb{R}^n$ with $\mathcal{H} = \{H_{ij}\}_{1\le i<j\le n}$, where $H_{ij} = \{x \in \mathbb{R}^n : x_i = x_j\}$.

Among the remarkably rich combinatorics of a (finite) Coxeter group is its poset structure, described as follows. Fix any connected component $\mathcal{K}$ in the complement $V \setminus (\bigcup_{H\in\mathcal{H}} H)$ of the hyperplanes. The connected component is an open cone whose closure has exactly $r = \dim V - \dim(\bigcap \mathcal{H})$ many facets. Let $S = \{s_1, \dots, s_r\} \subset W$ be the reflections by these facet hyperplanes, called *simple reflections*. Any $w \in W$ is a product of simple reflections; let $\ell(w)$ denote the minimum number for which one can write $w = s_{i_1} \cdots s_{i_{\ell(w)}}$, called the *length* of w.

Definition 10. The *Bruhat poset* $\mathcal{P}(W)$ is a poset on W defined as the transitive closure of the relation

$$w_1 < w_2 \quad \text{if and only if}$$
$$\ell(w_2) = \ell(w_1) + 1 \text{ and } w_2 = s_H w_1 \text{ for some } H \in \mathcal{H}.$$

Two facts about the Bruhat poset follow:

- It has a unique minimal element $\hat{0}$ (which is the identity Id) and a unique maximal element, called the *longest element* and denoted w_0. Let $d = \ell(w_0)$.
- It is a graded poset, graded by the length ℓ. That is, for all $w \in W$, every maximal chain in the interval $[\mathrm{Id}, w]$ has $\ell(w) + 1$ elements.

Remark 11. The group W acts freely and transitively on the set of connected components of the hyperplane arrangement complement $V \setminus (\bigcup \mathcal{H})$. We may thus label the components by W, with $\mathcal{K}$ labelled by the identity, and a component C by the unique $w \in W$ such that $C = w\cdot\mathcal{K}$. Then, the length $\ell(w)$ of $w \in W$ is the minimum number of hyperplanes that a path from $\mathcal{K}$ to the component $w \cdot \mathcal{K}$ must cross.

Example 12. Returning to the previous example of the symmetric group $\mathfrak{S}_n$ represented on $\mathbb{R}^n$, let $\mathcal{K} = \{x \in \mathbb{R}^n : x_1 > x_2 > \cdots > x_n\}$. The simple reflections $\{s_i\}_{1\le i\le n-1}$ correspond to adjacent transpositions $(i, i+1)$. Moreover, one can show, for instance using Remark 11, that the length of a permutation $w \in \mathfrak{S}_n$ is

$$\ell(w) = |\{(i,j) : 1 \leq i < j \leq n \text{ and } w(i) > w(j)\}|,$$

the number of inversions of w. The Bruhat poset of $\mathfrak{S}_4$ is illustrated in Figure 3.

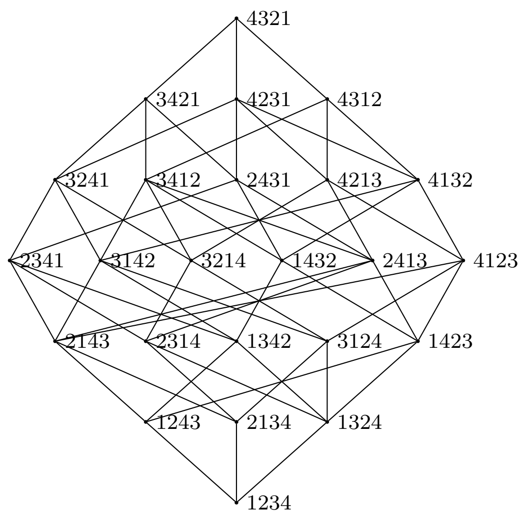


Figure 3: Diagram of the Bruhat poset $\mathcal{P}(\mathfrak{S}_4)$

(2) Questions. Given a graded poset $\mathcal{P}$ with a unique minimum $\hat{0}$, like the Bruhat poset $\mathcal{P}(W)$, we may consider its graded components. For $i \geq 0$, let

$$\mathcal{P}_i = \left\{ p \in \mathcal{P} : \begin{matrix} \text{a maximal chain } \hat{0} < \cdots < p \\ \text{consists of } i+1 \text{ elements} \end{matrix} \right\}$$

and let $p_i = |\mathcal{P}_i|$. A question of enduring interest for a graded poset concerns the *Sperner property*:

Is the maximum cardinality of an antichain (i.e., a subset consisting of pairwise incomparable elements) in $\mathcal{P}$ equal to the maximum of $\{p_0, \ldots, p_d\}$?

We may further ask whether $\mathcal{P}$ is *strongly Sperner*:

For each $0 \leq i \leq d/2$, is there a collection of p_i-many pairwise disjoint chains of the form $x_i < x_{i+1} < \cdots < x_{d-i}$ with $x_i \in \mathcal{P}_i$ and $x_{d-i} \in \mathcal{P}_{d-i}$?

The reader may verify as an exercise that the stated strongly Sperner property indeed implies the Sperner property. Note also that the strongly Sperner property implies that $(p_0, \ldots, p_d)$ is symmetric and unimodal. We explain how these questions can be answered in the case where $\mathcal{P} = \mathcal{P}(W)$, a Bruhat poset, by considering a graded algebra constructed from W.

(3) Algebras. Let $\mathrm{Sym}^\bullet(V)$ be the graded algebra of polynomials generated by a basis of V. The group W acts on it via its action on V, so we may consider the ideal I_+^W generated by the positive degree W-invariant polynomials

$$\{f \in \mathrm{Sym}^{>0}(V) : w \cdot f = f \quad \text{for all } w \in W\}.$$

Definition 13. The *coinvariant algebra* of W is the quotient

$$A^\bullet(W) = \mathrm{Sym}^\bullet(V)/I_+^W.$$

One finds the following properties of the coinvariant algebra from results of Bernstein, Gelfand, and Gelfand [BGG73]:

(i) There exists a collection of polynomials $\{P_w \in \mathrm{Sym}^\bullet(V) : w \in W\}$ that forms a vector space basis in $A^\bullet(W)$ satisfying $\deg P_w = \ell(w)$ for all $w \in W$. We call this basis the *Schubert basis* of $A^\bullet(W)$.[4]

(ii) Abusing notation, we write $\mathcal{K}$ also for the image in $A^1(W)$ of the connected region $\mathcal{K} \subset V$. This image is a nonempty open cone. For $L \in \mathcal{K}$, the Schubert basis $\{P_w\}$ of $A^\bullet(W)$ satisfies the property that $L \cdot P_w$ is a linear combination of $\{P_u : u \geq w\}$.

By property (i), we may choose an isomorphism $\deg \colon A^d(W) \xrightarrow{\sim} \mathbb{R}$ such that $\deg(P_{w_0}) > 0$.

Theorem 14 ([EW14])**.** The triple $(A^\bullet(W), \mathcal{K}, \deg)$ satisfies the Kähler package.

Combining the property (i) with the hard Lefschetz property of $A^\bullet(W)$ implies that $(p_0, \ldots, p_d)$

[4]When W is a Weyl group, these polynomials represents duals to the basis for the homology of the flag variety given by classes of Schubert varieties.

is symmetric and unimodal. Moreover, Stanley observed [Sta80b] that combining property (ii) with the hard Lefschetz property leads to the strongly Sperner property of $\mathcal{P} = \mathcal{P}(W)$ as follows.

Let $L \in \mathcal{K}$. For each $0 \le j \le d$, the Schubert basis realizes multiplication by $A^j(W) \xrightarrow{\cdot L} A^{j+1}(W)$ as a $p_{j+1} \times p_j$ matrix, denoted $L^{(j)}$. Denote by $L^{(j)}_{I,J}$ the submatrix of $L^{(j)}$ for a subset I of the rows and a subset J of the columns. Then, for $0 \le i \le d$, the Cauchy–Binet formula gives

$$\det(L^{(d-i-1)} \cdots L^{(i+1)} L^{(i)}) = \sum_{I_i,\dots,I_{d-i}} \prod_{j=i}^{d-i-1} \det L^{(j)}_{I_{j+1},I_j} \qquad (\dagger)$$

where the summation is over all sequences of subsets such that $p_i = |I_i| = \cdots = |I_{d-i}|$. As $L^{d-2i}\colon A^i(W) \to A^{d-i}(W)$ is an isomorphism, the left-hand-side in (†) is non-zero, so that a summand in the right-hand-side is non-zero for some $I_i, \dots, I_{d-i}$. In particular, each square matrix $L^{(j)}_{I_j,I_{j+1}}$ has a permutation pattern with all non-zero entries, which gives a sequence of matchings between $\mathcal{P}_j$ and $\mathcal{P}_{j+1}$ for all j. That these matchings respect the order on $\mathcal{P}$ follows from the property (ii).

Example 15. When $W = \mathfrak{S}_n$, we have $\mathrm{Sym}^\bullet(\mathbb{R}^n) = \mathbb{R}[x_1, \dots, x_n]$. A particular set of representatives for the Schubert basis of $A^\bullet(\mathfrak{S}_n)$, called *Schubert polynomials*, is given by the following rule: first, one sets $P_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$, and whenever $w(i) > w(i+1)$, one sets $P_{ws_i} = \partial_i P_w$ where ∂_i is an operator on $\mathbb{R}[x_1, \dots, x_n]$ defined by $\partial_i f = \frac{f - s_i \cdot f}{x_i - x_{i+1}}$. For example, when $n = 3$, we have

$$A^\bullet(\mathfrak{S}_3) = \frac{\mathbb{R}[x_1, x_2, x_3]}{\langle x_1 + x_2 + x_3, x_1x_2 + x_2x_3 + x_1x_3, x_1x_2x_3 \rangle},$$

and the elements $\{1, x_1, x_1 + x_2, x_1^2, x_1x_2, x_1^2x_2\}$ of $A^\bullet(\mathfrak{S}_3)$ form the Schubert basis.

Remark 16. Using similar methods, the results here for the Bruhat poset $\mathcal{P}(W)$ can be extended to the Bruhat poset $\mathcal{P}(W/W_J)$ on the set of cosets of a parabolic subgroup W_J of W. For classical types (A, B, C, and D), the strongly Sperner property of $\mathcal{P}(W)$ admits an elementary proof [Sta80b, Section 7], but this elementary method does not extend to the parabolic case $\mathcal{P}(W/W_J)$.

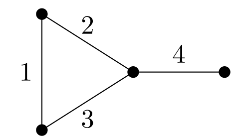
(4) Geometric origin.

When W satisfies an integrality condition of being *crystallographic*, it is the Weyl group of a complex semisimple Lie group G. In this case, Borel showed that the coinvariant algebra $A^\bullet(W)$ is the cohomology ring of the flag variety G/B, which is a smooth projective variety. Thus, the Kähler package for $A^\bullet(W)$ follows from classical Hodge theory in this case. The Schubert basis for $A^\bullet(W)$ comes from the *Bruhat decomposition* of G/B; it is a stratification of G/B into affine cells, whose closures in G/B thereby defines a basis for the cohomology ring. Bernstein–Gelfand–Gelfand showed how to interpret this geometric basis for $A^\bullet(W)$ purely algebraically.

For non-crystallographic finite Coxeter groups, where a Lie group G is no longer available, Elias and Williamson established the Kähler package in a much more general setting of Soergel bimodules, which we discuss in the next part.

(5) "Singular" objects. For an element $w \in W$, we consider the interval $[\mathrm{Id}, w]$ of $\mathcal{P}(W)$. Let us denote $\mathcal{P} = [\mathrm{Id}, w]$. We may ask similar questions for the interval $\mathcal{P}$ as we did for the whole poset $\mathcal{P}(W)$. However, a graded algebra with a basis labelled by the elements of the interval cannot in general satisfy the Kähler package — the cardinalities of the graded components of $[\mathrm{Id}, w]$ are often not even symmetric. For example, when $W = \mathfrak{S}_4$, the sequence $(p_0, \dots, p_4)$ for the interval $[\mathrm{Id}, 3412]$ reads $(1, 3, 5, 4, 1)$.

Geometrically, when W is the Weyl group of a Lie group G, this failure is reflected in the fact that the variety under concern is the *Schubert variety* X_w of $w \in W$, which is in general singular. This singular variety admits a particular desingularization, known as the *Bott–Samelson resolution*. Extracting key algebraic and combinatorial features of this desingularization leads to *Soergel bimodules*, which Elias and Williamson showed satisfy the Kähler package as formulated in Remark 3. Combining this with observations made by Björner and Ekedahl [BE09] (originally made only for Weyl groups), one can show that the interval $[\mathrm{Id}, w]$ satisfies a weak version of the Sperner property: for every $0 \le i \le \frac{\ell(w)}{2}$, there exists an injection $m\colon \mathcal{P}_i \hookrightarrow \mathcal{P}_{\ell(w)-i}$ such that $w \le m(w)$.



$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 4: A finite graph G and a matrix representing a subspace of $\Bbbk^4$ by its row span. The matroids associated with each are identical.

4 Matroids

(1) Objects. Let $E = \{1, 2, \ldots, n\}$ be a finite set.

Definition 17. A *matroid* M on E consists of a nonempty set $\mathcal{I}$ of subsets of E, called the *independent sets* of M, satisfying two properties:

- if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and
- if $I, J \in \mathcal{I}$ with $|I| < |J|$, then $I \cup \{j\} \in \mathcal{I}$ for some $j \in J \setminus I$.

Maximal elements of $\mathcal{I}$ have a common cardinality, which is called the *rank* r of M. To avoid trivialities, we suppose that M is loopless, i.e., that every one-element subset is independent. We point to [Wel71] as a general reference on matroids.

Example 18. Let G be a finite graph with edges E, such as the one illustrated in Figure 4. Then, the set $\mathcal{I} = \{$subsets of E which do not contain cycles$\}$ is the set of independent sets of a matroid. Such matroids are called *graphical matroids.*

Example 19. Let $L \subseteq \Bbbk^E$ be a vector subspace over a field $\Bbbk$. Then, the set $\mathcal{I} = \{I \subseteq E :$ the composition $L \hookrightarrow \Bbbk^E \twoheadrightarrow \Bbbk^I$ is surjective$\}$ is the set of independent sets of a matroid. Concretely, if L is the row span of an r by E matrix, then $\mathcal{I}$ consists of linearly independent subsets of the column vectors $\{v_e : e \in E\}$. Such matroids are called *linear* or *realizable matroids.*

We caution that while graphs and linear subspaces are two prototypical sources of matroids, almost every matroid (in some precise sense) does not arise in that way.

(2) Questions. We consider two sequences of numerical invariants of a matroid M of rank r. First, let $(I_0(\mathrm{M}), I_1(\mathrm{M}), \ldots, I_r(\mathrm{M}))$ be the sequence that counts the number of independent sets according to their cardinality, that is,

$$I_k(\mathrm{M}) = |\{I \in \mathcal{I} : |I| = k\}|.$$

For the second sequence, define the *rank function* rk_M by $\mathrm{rk}_\mathrm{M}(S) = \max\{|I| : I \subseteq S \text{ and } I \in \mathcal{I}\}$ for a subset $S \subseteq E$. Then, we define $(w_0(\mathrm{M}), \ldots, w_r(\mathrm{M}))$ by

$$w_k(\mathrm{M}) = \sum_{\substack{S \subseteq E \text{ such that} \\ \mathrm{rk}_\mathrm{M}(S) = k}} (-1)^{|S|+k},$$

which counts the subsets of the same rank with signs according to their cardinality. This sequence arises naturally in graph theory as follows. Let M be the matroid of a finite connected graph G with edges E on $(r+1)$ vertices. A *proper coloring* of G is a coloring of its vertices such that no incident vertices share the same color. Then, its *chromatic polynomial*

$$\chi_G(q) = \begin{array}{c}\text{the number of proper colorings} \\ \text{of } G \text{ with at most } q \text{ colors}\end{array}$$

is related to the sequence $(w_k(\mathrm{M}))$ by

$$\frac{\chi_G(q)}{q} = w_0(\mathrm{M}) q^r - w_1(\mathrm{M}) q^{r-1} + \cdots + (-1)^r w_r(\mathrm{M}).$$

Example 20. The reader may check that in the matroid of the graph in Figure 4, we have $(I_0(\mathrm{M}), I_1(\mathrm{M}), I_2(\mathrm{M}), I_3(\mathrm{M})) = (1, 4, 6, 3)$ and $(w_0(\mathrm{M}), w_1(\mathrm{M}), w_2(\mathrm{M}), w_3(\mathrm{M})) = (1, 4, 5, 2)$.

Long-standing conjectures from the 70's by various matroid theorists, including Mason, Rota, Heron, and Welsh, stated the following.

Both of the sequences $(I_0(\mathrm{M}),\dots,I_r(\mathrm{M}))$ *and* $(w_0(\mathrm{M}),\dots,w_r(\mathrm{M}))$ *are log-concave with no internal zeros.*

Both (and more related conjectures) were recently resolved by establishing the Kähler package for some algebras related to matroids, as we will now describe.

(3) Algebras. To define the algebra associated with a matroid M, we need the following notion: a subset $F \subseteq E$ is said to be a *flat* of M if it is a maximal subset of E of given rank, i.e., if $\mathrm{rk}_{\mathrm{M}}(F \cup e) > \mathrm{rk}_{\mathrm{M}}(F)$ for all $e \in E \setminus F$.

Definition 21. The *Chow ring of a matroid* M (with $\mathbb{R}$-coefficients) is the graded $\mathbb{R}$-algebra $A^\bullet(\mathrm{M}) = \bigoplus_{i=0}^{r-1} A^i(\mathrm{M})$ defined as the following quotient of a standard graded polynomial ring

$$A^\bullet(\mathrm{M}) = \frac{\mathbb{R}[x_F : F \text{ a non-empty flat of M}]}{I+J}$$

where $I = \langle x_F x_{F'} : F \not\subseteq F' \text{ and } F' \not\subseteq F\rangle$ and $J = \langle \sum_{F \ni i} x_F : i \in E\rangle$.

Let $\mathcal{K}$ be the set of elements $\xi \in A^1(\mathrm{M})$ that can be written as

$$\xi = \sum_{\emptyset \subsetneq F \subsetneq E} c(F) x_F$$

for some function $c\colon 2^E \to \mathbb{R}$ satisfying $c(\emptyset) = c(E) = 0$ and $c(S_1) + c(S_2) > c(S_1 \cap S_2) + c(S_1 \cup S_2)$ for all $S_1, S_2 \subseteq E$. Since the sum of two such functions retains this property, the set $\mathcal{K}$ is an open convex cone in $A^1(\mathrm{M})$ which is nonempty (for instance, let $c(S) = |S|\cdot|E \setminus S|$)). A recent breakthrough in matroid theory by Adiprasito, Huh, and Katz states that the Chow ring of a matroid satisfies the Kähler package.

Theorem 22 ([AHK18]). There is an isomorphism $\deg_{\mathrm{M}}\colon A^{r-1}(\mathrm{M}) \to \mathbb{R}$ defined by the property that $\deg((-x_E)^{r-1}) = 1$. The triple $(A^\bullet(\mathrm{M}), \mathcal{K}, \deg_{\mathrm{M}})$ satisfies the Kähler package.

To use the Kähler package of $A^\bullet(\mathrm{M})$ to deduce properties about the sequence $(w_0(\mathrm{M}),\dots,w_r(\mathrm{M}))$, we need the following theorem of Huh and Katz. Let $(\overline{w}_0(\mathrm{M}),\dots,\overline{w}_{r-1}(\mathrm{M}))$ be the sequence defined by the property $w_i(\mathrm{M}) = \overline{w}_i(\mathrm{M}) + \overline{w}_{i-1}(\mathrm{M})$ for all $i = 0,\dots,r$ (with $\overline{w}_{-1}(\mathrm{M}) = \overline{w}_r(\mathrm{M}) = 0$).

Theorem 23. Let α and β be elements of $A^1(\mathrm{M})$ defined by $\alpha = -x_E$ and $\beta = \sum_{\emptyset \subsetneq F \subseteq E} x_F$. Then, for $i = 0,\dots,r-1$, we have that

$$\deg_{\mathrm{M}}(\alpha^{r-1-i}\beta^i) = \overline{w}_i(\mathrm{M}).$$

Because the elements α and β are in the closure of $\mathcal{K}$, we thus conclude that $(\overline{w}_0(\mathrm{M}),\dots,\overline{w}_{r-1}(\mathrm{M}))$ is log-concave with no internal zeros. The relation $w_i(\mathrm{M}) = \overline{w}_i(\mathrm{M}) + \overline{w}_{i-1}(\mathrm{M})$ then easily implies that $(w_0(\mathrm{M}),\dots,w_r(\mathrm{M}))$ is log-concave with no internal zeros. Because the sequence $(I_0(\mathrm{M}),\dots,I_r(\mathrm{M}))$ can be realized as the sequence $(\overline{w}_0(\mathrm{M}'),\dots,\overline{w}_r(\mathrm{M}'))$ where M' is the *free co-extension matroid* of M, it is also log-concave with no internal zeros.

(4) Geometric origin. When M is the matroid of a linear subspace $L \subseteq \mathbb{C}^E$, the algebra $A^\bullet(\mathrm{M})$ is the cohomology ring of a complex smooth projective variety known as the *wonderful compactification* W_L of a hyperplane arrangement complement, introduced by De Concini and Procesi. Thus, in this case, the Kähler package for $A^\bullet(\mathrm{M})$ follows from classical Hodge theory.

Moreover, let us indicate a geometric origin for the formula $\deg_{\mathrm{M}}(\alpha^{r-1-i}\beta^i) = \overline{w}_i(\mathrm{M})$. Considered as elements in the cohomology ring of W_L, the elements α and β are divisor classes that define a map to a projective space. For β, the associated map $\varphi\colon W_L \to \mathbb{P}(\mathbb{C}^E)$ has the image known as the *reciprocal linear space*, whose degree was known to be $\overline{w}_{r-1}(\mathrm{M})$.

(5) "Singular" objects. Let us consider another numerical invariant of M. Define the sequence $(W_0(\mathrm{M}),\dots,W_r(\mathrm{M}))$ by

$$W_k(\mathrm{M}) = \text{the number of rank } k \text{ flats of M}.$$

Equivalently, let $\mathcal{P}(\mathrm{M})$ be the poset of flats of M ordered by inclusion, which is graded by rank. Let $\mathcal{P}_k(\mathrm{M})$ be the k-th graded component of $\mathcal{P}(\mathrm{M})$, i.e., the set of rank k flats of M, so that $W_k(\mathrm{M}) = |\mathcal{P}_k(\mathrm{M})|$.

It is conjectured but unknown whether this sequence is log-concave. Dowling and Wilson conjectured a related but different property, namely, that the sequence is *top-heavy*, i.e.

$$W_i(\mathrm{M}) \le W_{r-i}(\mathrm{M}) \text{ for all } i \le \tfrac{d}{2}.$$

More strongly, one may conjecture that there is an injection $m_i\colon \mathcal{P}_i(\mathrm{M}) \to \mathcal{P}_{r-i}(\mathrm{M})$ for each i such that $m_i(F) \supseteq F$ for all $F \in \mathcal{P}_i(\mathrm{M})$.

The resolution of this conjecture by Braden, Huh, Matherne, Proudfoot, and Wang [BHM$^+$] is featured in an upcoming *What is...?* AMS Notices survey. Let us give a rough sketch here.

As the question is about the graded poset $\mathcal{P}(\mathrm{M})$, taking an inspiration from the previous discussion about the strongly Sperner property for Bruhat posets (Section 3), one may seek an algebra whose basis is naturally labelled by the flats of M. Among possibly many such algebras, we consider the following one.

Definition 24. The *graded Möbius algebra* of a matroid M is a graded $\mathbb{R}$-algebra $B^\bullet(\mathrm{M}) = \bigoplus_{i=0}^r B^i(\mathrm{M})$ whose i-th graded component $B^i(\mathrm{M})$ is the vector space with basis $\{y_F : F \in \mathcal{P}_k(\mathrm{M})\}$, and with multiplication

$$y_F \cdot y_G = \begin{cases} y_{F\vee G} & \text{if } \mathrm{rk}_\mathrm{M}(F \vee G) = \mathrm{rk}_\mathrm{M}(F) + \mathrm{rk}_\mathrm{M}(G) \\ 0 & \text{otherwise,} \end{cases}$$

where $F \vee G$ denotes the unique minimal flat of M containing both F and G.

The graded Möbius algebra generally fails to have the Kähler package — the dimensions of its graded pieces are usually not even symmetric. Geometrically, this failure reflects that when M is the matroid of a linear subspace $L \subseteq \mathbb{C}^E$, the graded Möbius algebra is the cohomology ring of a singular projective complex variety called the *matroid Schubert variety*, which is the closure of L in $(\mathbb{CP}^1)^n$. This singular variety admits a particular desingularization, known as the *augmented wonderful compactification*. By extracting key algebraic and combinatorial features of this desingularization, one can construct the *intersection cohomology module* $IH^\bullet(\mathrm{M})$ of M, which is a graded $B^\bullet(\mathrm{M})$ module with an injection $B^\bullet(\mathrm{M}) \hookrightarrow IH^\bullet(\mathrm{M})$. Then, with significant effort, Braden, Huh, Matherne, Proudfoot, and Wang showed that $IH^\bullet(\mathrm{M})$ satisfies the Kähler package with $\mathcal{K}$ consisting of linear operators $\ell\colon IH^\bullet(\mathrm{M}) \to IH^{\bullet+1}(\mathrm{M})$ given by

$$\ell = \begin{array}{c} \text{multiplication by } \sum_{i\in E} a_i y_i \\ \text{where } a_i > 0 \text{ for all } i \in E. \end{array}$$

Then, since $B^\bullet(\mathrm{M})$ is a submodule of $IH^\bullet(\mathrm{M})$, using arguments similar to the one outlined in Section 3, one can conclude that there is an injection $m_i\colon \mathcal{P}_i(\mathrm{M}) \to \mathcal{P}_{r-i}(\mathrm{M})$ such that, for each i, $m_i(F) \supseteq F$. In particular, this proves the top-heavy conjecture of Dowling and Wilson.

5 Proving the Kähler package

It is usually very difficult to prove that an algebra has the Kähler package. For the cohomology ring of a complex Kähler manifold (or a complex projective manifold), the Kähler package is usually proved by analytic methods. However, analytic methods seem not suitable for the combinatorial settings described above in Sections 2, 3, and 4. In each of the three settings, classical Hodge theory can be used for a subset of the combinatorial objects that arise geometrically, but establishing the Kähler package in general requires an intricate analysis of combinatorics specific to each setting. Nonetheless, there is a basic inductive strategy common to all three settings, first introduced in the case of simplicial polytopes by McMullen [McM93]. It roughly consists of three steps:

1. One first proves (PD) by some direct argument. For example, often it is possible to compute a basis so that the matrix representing the pairing is upper triangular.

2. One then proves (HL) for all choices of elements of $\mathcal{K}$ by inductively assuming (HR) in "lower dimensions."

3. By (HL) and continuity, one then needs to verify (HR) for a single choice of an element in $\mathcal{K}$, often via another layer of induction.

We illustrate step 2 in the case of simplicial polytopes. Let P be a d-dimensional simplicial polytope in $\mathbb{R}^d$ which contains the origin in its interior. For each vertex v of P, let P_v be the polytope in $\mathbb{R}^d/(\mathbb{R}\cdot v)$

formed by taking the convex hull of the images of the vertices w such that $\{v, w\}$ is an edge of P. We may assume that the Kähler package holds for $A^\bullet(P_v)$ by induction on the dimension. There is a restriction map $\varphi_v \colon A^\bullet(P) \to A^\bullet(P_v)$ which has the property that $\varphi_v(\ell)$ is ample (as defined in Section 2) if $\ell \in A^1(P)$ is ample. When the degree map on $A^\bullet(P_v)$ is normalized appropriately, this map satisfies

$$\deg_P(a \cdot x_v) = \deg_{P_v}(\varphi_v(a)).$$

Since $\dim A^i(P) = \dim A^{d-i}(P)$ by (PD), in order to prove (HL) for $A^\bullet(P)$, it suffices to show that for each $i \leq d/2$ and $\ell_1, \ldots, \ell_{d-2i}$ ample, the map $A^i(P) \to A^{d-i}(P)$ given by multiplication by $\ell_1 \cdots \ell_{d-2i}$ is injective. Suppose $a \in A^i(P)$ is in the kernel of this map. Because ℓ_1 is ample, we can write $\ell_1 = \sum c_v x_v$, where $c_v > 0$ for each vertex v. We then have

$$\begin{aligned} 0 &= \deg_P(a^2 \ell_1 \cdots \ell_{d-2i}) \\ &= \sum_v c_v \deg_P(a^2 x_v \ell_2 \cdots \ell_{d-2i}) \\ &= \sum_v c_v \deg_{P_v}(\varphi_v(a)^2 \varphi_v(\ell_2) \cdots \varphi_v(\ell_{d-2i})). \end{aligned}$$

By construction, $\varphi_v(a)\varphi_v(\ell_1)\varphi_v(\ell_2) \cdots \varphi_v(\ell_{d-2i}) = 0$. Therefore, by (HR) for $A^\bullet(P_v)$, we have $(-1)^i \deg_{P_v}(\varphi_v(a)^2 \varphi_v(\ell_2) \cdots \varphi_v(\ell_{d-2i})) \geq 0$, with equality if and only if $\varphi_v(a) = 0$. We therefore must have $\varphi_v(a) = 0$ for all v. Then (PD) implies that $a = 0$.

After a variation of this argument is used to prove (HL) for $A^\bullet$, it remains to carry out step 3 and verify (HR). As the signature of a family of nondegenerate bilinear forms is constant, a continuity argument and (HL) for $A^\bullet$ shows that it suffices to verify the Hodge–Riemann relations for a single choice of ℓ_i. To do this, typically one finds a filtration

$$A_0^\bullet \subseteq A_1^\bullet \subseteq \cdots \subseteq A_k^\bullet = A^\bullet,$$

where each $A_i^\bullet$ is equipped with a cone $\mathcal{K}_i$ and an isomorphism $\deg_i \colon A_i^d \to \mathbb{R}$. The ring $A_0^\bullet$ is very simple, and one can verify by hand that (HR) holds for it. One attempts to prove that the Kähler package holds for $A_i^\bullet$ by induction on i. We typically have $\mathcal{K}_i \subset \overline{\mathcal{K}}_{i+1}$ and $A_{i+1}^\bullet \xrightarrow{\sim} A_i^\bullet \oplus (A_{i+1}^\bullet / A_i^\bullet)$ as $A_i^\bullet$-modules. Often this direct sum decomposition can be chosen to be orthogonal with respect to the bilinear form $(a, b) \mapsto \deg_{i+1}(a \cdot \ell^{d-2k} \cdot b)$ on A_{i+1}^k, for any $\ell \in \mathcal{K}_i$. One understands the signature of this bilinear form on $A_i^\bullet$ by induction, and one attempts to relate the module $A_{i+1}^\bullet / A_i^\bullet$ to objects of lower "dimension." Using this, one attempts to show that the restriction of this bilinear form to $A_{i+1}^\bullet / A_i^\bullet$ is nondegenerate and has the correct signature, verifying (HR) for ℓ.

In the case of polytopes one uses a version of the *weak factorization* theorem, which states that one can obtain any simplicial polytope from the simplex by a series of simple combinatorial operations. One verifies that (HR) is preserved under these operations. This reduces to verifying (HR) for the case of a simplex, which can be done directly.

Lastly, in Sections 2, 3, and 4, we mentioned that one can extend the validity of the Kähler package to "singular cases." An inductive strategy for doing so in the case of complex projective varieties can be found in [dCM05].

Acknowledgements. We thank Ben Elias, Eric Katz, Jacob Matherne, Chris McDaniel, Han–Bom Moon, Richard Stanley, and the referees for helpful comments on an earlier version of this article.

References

[AHK18] Karim Adiprasito, June Huh, and Eric Katz, *Hodge theory for combinatorial geometries*, Ann. of Math. (2) **188** (2018), no. 2, 381–452. MR3862944

[BE09] Anders Björner and Torsten Ekedahl, *On the shape of Bruhat intervals*, Ann. of Math. (2) **170** (2009), no. 2, 799–817. MR2552108

[BGG73] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand, *Schubert cells, and the cohomology of the spaces G/P*, Uspehi Mat. Nauk **28** (1973), no. 3(171), 3–26. MR429933

[BHM+] Tom Braden, June Huh, Jacob Matherne, Nicholas Proudfoot, and Botong Wang, *Singular Hodge theory for combinatorial geometries*. arXiv:2010.06088.

[BL80] Louis J. Billera and Carl W. Lee, *Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), no. 1, 181–185. MR551759

[dCM05] Mark Andrea A. de Cataldo and Luca Migliorini, *The Hodge theory of algebraic maps*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 5, 693–750. MR2195257

[DN06] Tien-Cuong Dinh and Viêt-Anh Nguyên, *The mixed Hodge-Riemann bilinear relations for compact Kähler manifolds*, Geom. Funct. Anal. **16** (2006), no. 4, 838–849. MR2255382

[Eur24] Christopher Eur, *Essence of independence: Hodge theory of matroids since June Huh*, Bull. Amer. Math. Soc. (N.S.) **61** (2024), no. 1, 73–102. MR4678572

[EW14] Ben Elias and Geordie Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) **180** (2014), no. 3, 1089–1136. MR3245013

[HMM+13] Tadahito Harima, Toshiaki Maeno, Hideaki Morita, Yasuhide Numata, Akihito Wachi, and Junzo Watanabe, *The Lefschetz properties*, Lecture Notes in Mathematics, vol. 2080, Springer, Heidelberg, 2013. MR3112920

[Huh23] June Huh, *Combinatorics and Hodge theory*, ICM—International Congress of Mathematicians. Vol. I. Prize lectures, 2023, pp. 212–239. MR4680249

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460

[Kar04] Kalle Karu, *Hard Lefschetz theorem for nonrational polytopes*, Invent. Math. **157** (2004), no. 2, 419–447. MR2076929

[McM93] Peter McMullen, *On simple polytopes*, Invent. Math. **113** (1993), no. 2, 419–444. MR1228132

[Sta75] Richard P. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*, Studies in Appl. Math. **54** (1975), no. 2, 135–142. MR458437

[Sta80a] ———, *The number of faces of a simplicial convex polytope*, Adv. in Math. **35** (1980), no. 3, 236–238. MR563925

[Sta80b] ———, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980), no. 2, 168–184. MR578321

[Sta87] ———, *Generalized H-vectors, intersection cohomology of toric varieties, and related results*, Commutative algebra and combinatorics (Kyoto, 1985), 1987, pp. 187–213. MR951205

[Wel71] D. J. A. Welsh, *Combinatorial problems in matroid theory*, Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969), 1971, pp. 291–306. MR0278975

[Zie95] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR1311028