MATH 54 FALL 2016: DISCUSSION 102/105 QUIZ#7

GSI: CHRISTOPHER EUR, DATE: 10/14/2016

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Problem 1. (4 points) Suppose $A\vec{x} = \vec{b}$ is an *inconsistent* system of equations. Then show that

$$\operatorname{rank} A < \operatorname{rank}[A|\vec{b}]$$

where $[A|\vec{b}]$ is the matrix A augmented by a column $\vec{b}$.

Problem 2. Let $\mathcal{P}_2 := \{a_2x^2 + a_1x + a_0 : a_2, a_1, a_0 \in \mathbb{R}\}$ be the vector space of polynomials of degree at most 2. Let $B = (x^2, x, 1)$ be a basis of $\mathcal{P}_2$. Consider a linear map $T : \mathcal{P}_2 \to \mathbb{R}$ given by

$$T(p(x)) = \int_0^1 p(t)dt$$

(a) (2 points) Write down the matrix A of the linear map T with respect to the basis B on $\mathcal{P}_2$ and the standard basis on $\mathbb{R}$.
(b) (1 point) Verify that $\int_0^1 x^2 + 2x = \frac{4}{3}$ by multiplying A to the coordinate vector corresponding to $x^2 + 2x$ (w/r/t basis B) .
(c) (2 points) Find a basis for $\ker T$, and a basis for $\operatorname{im} T$.
(d) (1 point) Verify the rank-nullity theorem for this linear map T.

1. Let $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$. $A\vec{x} = \vec{b}$ inconsistent means $\vec{b} \notin \operatorname{span}(a_1, \dots, a_n)$.

Hence, $\operatorname{span}(a_1, \dots, a_n)$ is strictly smaller than $\operatorname{span}(a_1, \dots, a_n, b)$

Thus, $\operatorname{rk} A = \dim \operatorname{span}(a_1, \dots, a_n) < \dim \operatorname{span}(a_1, \dots, a_n, b) = \operatorname{rk}[A|b]$.

2. (a) $\int_0^1 x^2 dx = \frac{1}{3}$, $\int_0^1 x\, dx = \frac{1}{2}$, $\int_0^1 1\, dx = 1$. Hence,

$$\boxed{\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}}$$

(b)

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{3} + 1 = \frac{4}{3} \quad \checkmark$$

(c) $A \neq 0 \Rightarrow \operatorname{rk} A \geq 1 \Rightarrow \operatorname{im} T = \mathbb{R}$ (as $\dim \mathbb{R} = 1$). $\boxed{\operatorname{im} T = \operatorname{span}(1).}$

kernel: $\operatorname{null} A = \operatorname{span}\left(\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right)$. So, $\boxed{\ker T = \operatorname{span}(-3x^2+1, -2x+1)}$

(d) $\dim \ker T = 2$ (see (c)).
$\dim \operatorname{im} T = 1$ (") and $\dim \mathcal{P}_2 = 3$. So

$$\begin{matrix} \dim \mathcal{P}_2 & = & \dim \ker T & + & \dim \operatorname{im} T \\ \| & & \| & & \| \\ 3 & & 2 & + & 1 \end{matrix} \quad \checkmark$$