MATH 54 FALL 2016: DISCUSSION 102/105 QUIZ#11

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Problem 1. Define an inner product on $\mathcal{P}_2$ (polynomials in t of degree ≤ 2) by

$$\langle f(t), g(t)\rangle := \int_0^1 f(t)g(t)dt$$

(a) (1 point) Find the orthogonal basis for the subspace $W := \text{span}(1, t)$ of $\mathcal{P}_2$.
(b) (3 points) Find the polynomial $p(t)$ **of degree $\leq$1** that minimizes the quantity

$$\int_0^1 \left(t^2 + t + 1 - p(t)\right)^2 dt$$

Problem 2. True/False: (2 points) Assume that all the following are matrices with real coefficients.
(a) An orthogonally diagonalizable matrix is symmetric.
(b) An orthogonal matrix is orthogonally diagonalizable.
(c) If $(v_1, \ldots, v_n)$ is an eigenbasis of a symmetric matrix M, then it is also an orthogonal basis.

#1.(a) $w_1 = 1$, $w_2 = t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle}1 = t - \frac{1}{2}$. $\boxed{(1,\ t-\tfrac{1}{2})}$ (by Gram-Schmidt)

#1 (b) Minimize $\langle t^2+t+1-p,\ t^2+t+1-p\rangle$ where $p \in W = \text{span}(1,t)$ $\Leftrightarrow$ $p(t) = \text{Proj}_{\text{span}(1,t)}\, t^2+t+1$

Well, $\text{Proj}_{\text{span}(1,t)}(t^2+t+1) = \dfrac{\langle t^2+t+1, 1\rangle}{\langle 1, 1\rangle}1 + \dfrac{\langle t^2+t+1,\ t-\frac{1}{2}\rangle}{\langle t-\frac{1}{2},\ t-\frac{1}{2}\rangle}(t-\tfrac{1}{2})$

$\int_0^1 t^3 + \frac{1}{2}t^2 + \frac{1}{2}t - \frac{1}{2}\, dt$

$\int_0^1 t^2 - t + \frac{1}{4}\, dt$

$$= \left(\tfrac{1}{3}+\tfrac{1}{2}+1\right)1 + \frac{\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{4}-\frac{1}{2}\right)}{\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{4}\right)}(t-\tfrac{1}{2})$$

$$= \frac{11}{6} + \frac{\frac{1}{6}}{\frac{1}{12}}(t-\tfrac{1}{2}) = \frac{11}{6} + 2t - 1 = \boxed{2t + \tfrac{5}{6}}$$

#2 (a) $PAP^T = D$ w/ $P^T = P^{-1}$, so $A = P^T D P$, thus $A^T = P^T D^T P^{TT} = P^T D P = A$ ✓ True.

(b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is not even diagonalizable... False

(c) $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then any basis is an eigen basis, e.g. $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. False