Transportation problem example, in detail

Warehouses A, B, and C have 18, 25, and 12 units of a certain commodity, respectively. Customers V, W, X, Y, and Z have respective demands of 14, 6, 8, 10, and 17 units of this commodity. The costs per unit to ship this commodity from each warehouse to each customer are given in the table below:

	V	W	X	Y	Z
A	\$20	\$12	\$15	\$10	\$16
B	\$15	\$18	\$16	\$13	\$10
C	\$17	\$14	\$19	\$14	\$12

How can all of these demands be met from these warehouses at minimum total cost?

We will represent a solution in the form of a 3×5 grid (or *tableau*), as shown below.

	V	W	X	Y	Z
A					
B					
C					

The three rows of this grid correspond to the warehouses A, B, and C, and the five columns correspond to the customers V, W, X, Y, and Z. We will write a bold number in the lower left corner of a square (or *cell*) in this grid to represent the number of units shipped from the corresponding warehouse to the corresponding customer; we might call such a quantity a *flow*. (We will also write other numbers in other locations in the square.)

We begin by finding an *initial basic feasible solution*. The purpose of this initial basic feasible solution is to give us a starting point—just one possible way of satisfying the demands from the warehouses. It will not be optimal; in fact, we will not even consider the shipping costs at this point.

The **method to find an initial basic feasible solution** is straightforward:

- Satisfy each demand one by one, from left to right.
- For each demand, take units from each supply, top to bottom, until the demand is met; but once a supply has been exhausted, it cannot be used to satisfy later demands.
- [Note: There is a special case that arises when a demand is satisfied *and* a supply is exhausted simultaneously. See the note and example on page 17 for details.]

For this problem, we form an initial basic feasible solution as follows:

0. For convenience, let's write the amount of each supply (i.e., each warehouse) next to the row labels, and the amount of each demand (i.e., each customer) next to the column labels.

	V: 14	W: 6	X: 8	Y: 10	Z: 17
A: 18					
B: 25					
C: 12					

1. The first demand is V, which demands 14 units. This demand can be satisfied by taking 14 units from the first supply, A. So we will do that.

	V: 14	W: 6	X: 8	Y: 10	Z: 17
A: 18	**14**				
B: 25					
C: 12					

2. The next demand is W, which demands 6 units. Warehouse A has 4 units remaining (because 14 of its 18 units were previously used to satisfy the demand of V), so W will take those remaining 4 units from A; this exhausts the supply of A. The remaining 2 units of W's demand can be satisfied by taking 2 units from the next supply, B.

	V: 14	W: 6	X: 8	Y: 10	Z: 17
A: 18	**14**	**4**			
B: 25		**2**			
C: 12					

3. The third demand is X, which demands 8 units. Warehouse B has 23 units remaining (after W took 2 units), so we can satisfy the demand of X by taking 8 units from B.

	V: 14	W: 6	X: 8	Y: 10	Z: 17
A: 18	**14**	**4**			
B: 25		**2**	**8**		
C: 12					

4. The fourth demand, Y, is for 10 units. We can satisfy this demand by taking 10 units from B.

	V: 14	W: 6	X: 8	Y: 10	Z: 17
A: 18	**14**	**4**			
B: 25		**2**	**8**	**10**	
C: 12					

5. The last demand is Z, which demands 17 units. The second supply, B, has 5 units remaining (after W took 2, X took 8, and Y took 10), so Z will take those remaining 5 units from B (after which B is exhausted), and then we will satisfy the remaining 12 units of Z's demand by taking all 12 units from the last supply, C.

	V: 14	W: 6	X: 8	Y: 10	Z: 17
A: 18	**14**	**4**			
B: 25		**2**	**8**	**10**	**5**
C: 12					**12**

This process gives us an initial basic feasible solution. The squares containing bold numbers here are called *basic* squares; they correspond to basic variables in the linear programming formulation of this problem (the LP formulation has 15 variables, one for each square). The squares not containing bold numbers are *nonbasic*. Note that the basic squares form a "stair-step" pattern from the upper left square of the grid to the lower right square; this should always be the case. (See the note and example on page 17 for a discussion about a special case that occasionally arises.)

This initial solution is *feasible* because all of the demands are met, and all of the supplies are exhausted. (Notice that the total supply is equal to the total demand in this problem. If this is not the case, then you will need to add a fictitious supply or a fictitious demand to make the totals equal.) The initial solution is *basic* because the number of basic squares is equal to the number of rows plus the number of columns minus 1 (which is $3 + 5 - 1 = 7$ for this problem); that is the rank of the coefficient matrix of the LP for this problem.

Important: Every solution that we get during the solution process must be basic. In other words, the number of basic squares must always be equal to the number of rows plus the number of columns minus 1.

(In this problem, with a 3×5 grid, that means we must always have $3 + 5 - 1 = 7$ basic squares. We achieved this result with our initial basic feasible solution, but we will see where it is important to take this rule into consideration a little later in this example—see page 15.)

But there is no reason that the initial basic feasible solution should be *optimal*, because we didn't consider the costs at all. The initial basic feasible solution just gives us a starting point. It's probably a terribly inefficient solution to the problem; it probably costs much more than the optimal solution does. The rest of the algorithm for solving this transportation problem consists of repeatedly *improving* the current solution (lowering its cost) until it cannot be improved further, at which point the solution is optimal.

A key step in this improvement process is identifying the values of the dual variables. There are 15 variables in the primal LP, corresponding to the squares in the grid, and representing the number of units to be shipped from the corresponding warehouse to the corresponding customer. The primal LP has eight constraints: three to ensure that each supply is exhausted (these correspond to the rows of the grid), and five to ensure that each demand is met (these correspond to the columns of the grid). So the dual LP will have eight variables: one for each row and one for each column. Let's call the "row variables" v_1, v_2, and v_3, and let's call the "column variables" w_1 through w_5. We can use complementary slackness to determine the values of the dual variables from the values of the primal variables, which we have in the grid. From complementary slackness, if the primal solution is optimal, then basic primal variables should correspond to tight dual constraints, which gives us the following:

Rule from complementary slackness: For each **basic** square in the grid, the sum of the corresponding row variable and the corresponding column variable must equal the corresponding cost.

Now, in this problem we have eight dual variables (three row variables and five column variables), but only seven basic squares, which means that complementary slackness gives us only seven equations for these eight unknowns. That means that we have one *free variable*; we can choose one of the dual variables to be any value that we like, and then solve for the remaining dual variables using the rule above. (It will always be true that there is one more dual variable than basic squares, so there will always be one free variable.) In practice, for simplicity, we choose the value of the first row variable, v_1, to be 0, and go from there.

Let's see how we can find the values of the dual variables for our initial basic feasible solution. To assist us in this process, we will copy the costs from the table given in the problem into the grid. We'll put the costs in little postage-stamp boxes in the upper right corners of each square, like this:

	V	W	X	Y	Z
A	20 **14**	12 **4**	15	10	16
B	15	18 **2**	16 **8**	13 **10**	10 **5**
C	17	14	19	14	12 **12**

We'll write the row variables on the right-hand side of each row, and we'll write the column variables at the bottom of each column:

	V	W	X	Y	Z	
A	20 **14**	12 **4**	15	10	16	v_1
B	15	18 **2**	16 **8**	13 **10**	10 **5**	v_2
C	17	14	19	14	12 **12**	v_3
	w_1	w_2	w_3	w_4	w_5	

The first step in the process of finding the values of the dual variables is to set $v_1 = 0$, as our free variable:

	V	W	X	Y	Z	
A	20 **14**	12 **4**	15	10	16	$v_1 = 0$
B	15	18 **2**	16 **8**	13 **10**	10 **5**	v_2
C	17	14	19	14	12 **12**	v_3
	w_1	w_2	w_3	w_4	w_5	

Now, consider the AV square in this grid (that is, the square at the intersection of the A row and the V column). This is a basic square, so the result of complementary slackness says that the sum of the corresponding row variable v_1 and the corresponding column variable w_1 must equal the corresponding cost, which is 20; so we have $v_1 + w_1 = 20$. Since $v_1 = 0$, that means that w_1 must be 20:

	V	W	X	Y	Z	
A	20 **14**	12 **4**	15	10	16	$v_1 = 0$
B	15	18 **2**	16 **8**	13 **10**	10 **5**	v_2
C	17	14	19	14	12 **12**	v_3
	$w_1 = 20$	w_2	w_3	w_4	w_5	

Next, consider the AW square, which is also basic. Complementary slackness, when applied to this square, says that $v_1 + w_2 = 12$. Since $v_1 = 0$, that means that $w_2 = 12$:

	V	W	X	Y	Z	
A	20 **14**	12 **4**	15	10	16	$v_1 = 0$
B	15	18 **2**	16 **8**	13 **10**	10 **5**	v_2
C	17	14	19	14	12 **12**	v_3
	$w_1 = 20$	$w_2 = 12$	w_3	w_4	w_5	

Then we can consider the BW square. This is a basic square, so complementary slackness gives $v_2 + w_2 = 18$. Since we now know that $w_2 = 12$, this means that $v_2 = 6$:

	V	W	X	Y	Z	
A	(20) **14**	(12) **4**	(15)	(10)	(16)	$v_1 = 0$
B	(15)	(18) **2**	(16) **8**	(13) **10**	(10) **5**	$v_2 = 6$
C	(17)	(14)	(19)	(14)	(12) **12**	v_3
	$w_1 = 20$	$w_2 = 12$	w_3	w_4	w_5	

Next, consider the BX square. Because this square is basic, we get the equation $v_2+w_3 = 16$. Since $v_2 = 6$, we see that $w_3 = 10$:

	V	W	X	Y	Z	
A	(20) **14**	(12) **4**	(15)	(10)	(16)	$v_1 = 0$
B	(15)	(18) **2**	(16) **8**	(13) **10**	(10) **5**	$v_2 = 6$
C	(17)	(14)	(19)	(14)	(12) **12**	v_3
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	w_4	w_5	

From the BY square, which is basic, we get $v_2 + w_4 = 13$; since $v_2 = 6$, we get $w_4 = 7$:

	V	W	X	Y	Z	
A	(20) **14**	(12) **4**	(15)	(10)	(16)	$v_1 = 0$
B	(15)	(18) **2**	(16) **8**	(13) **10**	(10) **5**	$v_2 = 6$
C	(17)	(14)	(19)	(14)	(12) **12**	v_3
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	w_5	

From the BZ square, which is basic, we get $v_2 + w_5 = 10$; since $v_2 = 6$, we get $w_5 = 4$:

	V	W	X	Y	Z	
A	(20) **14**	(12) **4**	(15)	(10)	(16)	$v_1 = 0$
B	(15)	(18) **2**	(16) **8**	(13) **10**	(10) **5**	$v_2 = 6$
C	(17)	(14)	(19)	(14)	(12) **12**	v_3
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

Lastly, from the final basic square, CZ, we get $v_3 + w_5 = 12$; since $w_5 = 4$, we get $v_3 = 8$:

	V	W	X	Y	Z	
A	(20) **14**	(12) **4**	(15)	(10)	(16)	$v_1 = 0$
B	(15)	(18) **2**	(16) **8**	(13) **10**	(10) **5**	$v_2 = 6$
C	(17)	(14)	(19)	(14)	(12) **12**	$v_3 = 8$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

Once we have all of the values of the dual variables, we compute *test values* for all of the *nonbasic* squares in the grid.

> **Computing test values:** A test value (for a **nonbasic** square only) is equal to the cost of that square, minus the value of the corresponding row variable, minus the value of the corresponding column variable.

The meaning of a test value is as follows: A nonbasic square is a link from a supply to a demand that the current solution is not using. If the test value of that square is *negative*, then we can improve our current solution (to get a solution of lower cost) by using that link.

We will write test values in the lower right corners of nonbasic squares, and to further distinguish them from the values in the basic squares we will color them red. Note that test values have very different meanings from the values in basic squares! The values in basic squares (written in the lower left corner) represent *flows* from supplies to demands; they are quantities that are being shipped from the corresponding supply to the corresponding demand. The values in nonbasic squares (written in the lower right corner) are test values that tell us whether it would be advantageous to change our solution to use the corresponding link from a supply to a demand.

Let's compute the test values for the nonbasic squares in this example. The test values can be computed in any order; let's begin in the A row and work down. The first nonbasic square is the AX square. The test value for the AX square is the cost minus the row variable minus the column variable, which is $15 - 0 - 10 = 5$:

	V	W	X	Y	Z	
A	20 **14**	12 **4**	15 5	10	16	$v_1 = 0$
B	15	18 **2**	16 **8**	13 **10**	10 **5**	$v_2 = 6$
C	17	14	19	14	12 **12**	$v_3 = 8$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

The test value for the next nonbasic square, the AY square, is $10 - 0 - 7 = 3$:

	V	W	X	Y	Z	
A	20 **14**	12 **4**	15 5	10 3	16	$v_1 = 0$
B	15	18 **2**	16 **8**	13 **10**	10 **5**	$v_2 = 6$
C	17	14	19	14	12 **12**	$v_3 = 8$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

The test value for the AZ square is $16 - 0 - 4 = 12$:

	V	W	X	Y	Z	
A	20 **14**	12 **4**	15 5	10 3	16 12	$v_1 = 0$
B	15	18 **2**	16 **8**	13 **10**	10 **5**	$v_2 = 6$
C	17	14	19	14	12 **12**	$v_3 = 8$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

Starting the next row, the test value for the BV square is $15 - 6 - 20 = -11$:

	V	W	X	Y	Z	
A	[20] **14**	[12] **4**	[15] 5	[10] 3	[16] 12	$v_1 = 0$
B	[15] −11	[18] **2**	[16] **8**	[13] **10**	[10] **5**	$v_2 = 6$
C	[17]	[14]	[19]	[14]	[12] **12**	$v_3 = 8$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

That was the only nonbasic square in the B row, so we'll continue to the C row. The test value for the CV square is $17 - 8 - 20 = -11$:

	V	W	X	Y	Z	
A	[20] **14**	[12] **4**	[15] 5	[10] 3	[16] 12	$v_1 = 0$
B	[15] −11	[18] **2**	[16] **8**	[13] **10**	[10] **5**	$v_2 = 6$
C	[17] −11	[14]	[19]	[14]	[12] **12**	$v_3 = 8$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

The test value for the CW square is $14 - 8 - 12 = -6$:

	V	W	X	Y	Z	
A	[20] **14**	[12] **4**	[15] 5	[10] 3	[16] 12	$v_1 = 0$
B	[15] −11	[18] **2**	[16] **8**	[13] **10**	[10] **5**	$v_2 = 6$
C	[17] −11	[14] −6	[19]	[14]	[12] **12**	$v_3 = 8$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

The test value for the CX square is $19 - 8 - 10 = 1$:

	V	W	X	Y	Z	
A	[20] **14**	[12] **4**	[15] 5	[10] 3	[16] 12	$v_1 = 0$
B	[15] −11	[18] **2**	[16] **8**	[13] **10**	[10] **5**	$v_2 = 6$
C	[17] −11	[14] −6	[19] 1	[14]	[12] **12**	$v_3 = 8$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

And, finally, the test value for the CY square is $14 - 8 - 7 = -1$:

	V	W	X	Y	Z	
A	[20] **14**	[12] **4**	[15] 5	[10] 3	[16] 12	$v_1 = 0$
B	[15] −11	[18] **2**	[16] **8**	[13] **10**	[10] **5**	$v_2 = 6$
C	[17] −11	[14] −6	[19] 1	[14] −1	[12] **12**	$v_3 = 8$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 10$	$w_4 = 7$	$w_5 = 4$	

Since there are negative test values, we know that this solution is not optimal (which is not surprising, because we wrote down this solution without even considering the costs). We can improve the solution by shifting the flows in order to use one of the links that has a negative test value. As a rule of thumb, we will choose a link that has the *most* negative test value. Here, two nonbasic squares (BV and CV) tie for the most negative test value, so we can pick one of those squares arbitrarily. Let's pick the BV square. We will change our solution to use that link by changing the flows in the grid; this operation is called *pivoting* (it is equivalent to pivoting in the simplex algorithm).

Choosing a pivot: If there is a negative test value in the grid, then the solution is not optimal. In this case, choose the most negative test value as the pivot location. If there is a tie, break the tie arbitrarily.

We want to change our solution to use the BV link (i.e., to ship some units from the supply B to the demand V). We don't know yet how many units we should send along this link, so let's call that unknown quantity t.

	V	W	X	Y	Z
A					
B	$\boldsymbol{t}$				
C					

Now, consider the V column. The total demand of V is 14, which means that the total of the flows in that column should be 14. The current solution satisfies that demand by shipping 14 units from A to V. But now we are going to ship t units from B to V, so in order to maintain the same column sum we must *subtract* t units from the flow in the AV square, giving us a flow of $14 - t$ there:

	V	W	X	Y	Z
A	$\boldsymbol{14 - t}$				
B	$\boldsymbol{t}$				
C					

Next, consider the A row. The total supply of A is 18, which means that the total of the flows in that row should be 18. The current solution exhausts that supply by shipping 14 units from A to V and 4 units from A to W. But now we are reducing the flow in the AV square by t units, so we need to *increase* the flow in the AW square by t units in order to maintain the same row sum:

	V	W	X	Y	Z
A	$\boldsymbol{14 - t}$	$\boldsymbol{4 + t}$			
B	$\boldsymbol{t}$				
C					

And now consider the W column. The total demand of W is 6, which means that the total of the flows in that column should be 6. The current solution satisfies that demand by shipping 4 units from A to W and 2 units from B to W. But now we are going to increase the flow in the AW square by t units, so we need to *subtract* t units from the BW square in order to maintain the same column sum:

	V	W	X	Y	Z
A	$\mathbf{14-t}$	$\mathbf{4+t}$			
B	$\boldsymbol{t}$	$\mathbf{2-t}$			
C					

Note that this last adjustment also fixed the discrepancy in the B row, so that the row sum in that row will remain the same: we are increasing the flow in the BV square by t and also reducing the flow in the BW square by t, so the net change is zero. After these adjustments, all of the row sums and all of the column sums remain the same as they were before, which means that all of the demands will be satisfied and all of the supplies will be exhausted, so we will get another feasible solution. The other flows in the basic squares that were not adjusted will remain the same:

	V	W	X	Y	Z
A	$\mathbf{14-t}$	$\mathbf{4+t}$			
B	$\boldsymbol{t}$	$\mathbf{2-t}$	**8**	**10**	**5**
C					**12**

A useful way of viewing this series of adjustments is to view it as a *circuit* in the grid representing the previous solution. The circuit begins in the nonbasic square where we are pivoting (i.e., the BV square), and then travels alternately vertically and horizontally, "landing" only in basic squares (AV, AW, and BW) and then returning to the pivot square (BV). This circuit is shown below, superimposed on the grid representing the previous solution:

	V	W	X	Y	Z
A	[20] **14**	[12] **4**	[15] 5	[10] 3	[16] 12
B	[15] −11	[18] **2**	[16] **8**	[13] **10**	[10] **5**
C	[17] −11	[14] −6	[19] 1	[14] −1	[12] **12**

We can perform the necessary adjustments by identifying such a circuit (there will be exactly one such circuit in the grid) and then alternately adding and subtracting t from the flows in the squares going around the circuit, in either direction, starting at the pivot square.

These adjustments give us the following tentative new grid:

	V	W	X	Y	Z
A	$\mathbf{14-t}$	$\mathbf{4+t}$			
B	$\boldsymbol{t}$	$\mathbf{2-t}$	**8**	**10**	**5**
C					**12**

We would like to make the value of t as large as possible, because using the BV link saves us money (that's what the negative test value tells us). However, we can't make t too large, because none of the flows can become negative (we can't ship a negative number of units from a supply to a demand). So we need to choose $t = 2$ here, because one of the squares in the tentative new grid, the BW square, has a flow of $2 - t$. When we choose $t = 2$, the flow in the BW square becomes zero, so that square becomes nonbasic. We gain a new basic square (BV) and lose an old basic square (BW), so the number of basic squares remains the same; this is important, because the number of basic squares in this problem should always be 7. The result is the following basic feasible solution:

	V	W	X	Y	Z
A	**12**	**6**			
B	**2**		**8**	**10**	**5**
C					**12**

Note that the "stair-step" structure is gone. That's okay; that was only a requirement for the *initial* basic feasible solution.

The steps of pivoting are summarized below:

Pivoting: After choosing the pivot location in the grid, perform the following steps to pivot there.

1. Find a *circuit* in the grid, starting at the pivot square. The circuit must have the following properties:
   - It must begin and end at the pivot square.
   - From the pivot square, the circuit must go vertically to a basic square, then horizontally to another basic square, then vertically to another basic square, and so on, alternating vertical and horizontal steps, until it returns to the pivot square. These vertical and horizontal steps can (and usually will) jump over some squares in the grid.
   - Except for the pivot square, the vertical and horizontal steps of the circuit must always land in basic squares.
   - The circuit cannot land in the same square more than once, except the pivot square at the beginning and end of the circuit.

   There will be exactly one such circuit in the grid that begins and ends at the pivot square. There are no hard and fast rules for finding the circuit; you will have to do some searching (and perhaps some trial-and-error) to find it.
2. Form a tentative new grid by adding t to the flow of the pivot square (which was previously zero), and then alternately subtracting and adding t to the basic squares going around the circuit (in either direction) to preserve all of the row sums and column sums.
3. Choose the largest value of t that does not make any flow negative. Then determine the new flows by using this value of t. A flow that becomes zero will become a nonbasic square.
   - However, if more than one flow becomes zero, only one of them will become nonbasic, in order to preserve the correct number of basic squares. The others will become basic squares that happen to have zero flow (this is a degenerate solution). Choose any one of the zero-flow squares to become nonbasic, and leave the rest as basic squares with zero flow.

So now we have a new basic feasible solution. We will continue this process (finding the values of the dual variables, computing the test values, identifying a pivot location, and pivoting there) until we reach a point where all of the test values are nonnegative; this means that the solution is optimal, and we can stop. Therefore, the whole process of solving a transportation problem goes like this:

Steps for solving a transportation problem:

1. Form an initial basic feasible solution. See the box on page 1.
2. Write the costs in the grid, and set the first row variable, v_1, to 0.
3. Use the **basic** squares and the rule from complementary slackness to find the values of all the other dual variables (i.e., the row and column variables). See the box on page 3.
4. Compute the test values for all of the **nonbasic** squares. See the box on page 6.
5. If all of the test values are nonnegative, then the current solution is optimal. Stop.
6. Otherwise, choose the most negative test value and pivot there (by finding a circuit and adding and subtracting t). See the box on page 10.
7. Return to step 2 for the new solution obtained in step 6.

Let's continue with our example. We continue by adding the costs to the upper right corners of the squares and setting $v_1 = 0$:

	V	W	X	Y	Z	
A	[20] **12**	[12] **6**	[15]	[10]	[16]	$v_1 = 0$
B	[15] **2**	[18]	[16] **8**	[13] **10**	[10] **5**	v_2
C	[17]	[14]	[19]	[14]	[12] **12**	v_3
	w_1	w_2	w_3	w_4	w_5	

Now we use the rule from complementary slackness to find the values of the remaining dual variables.

1. The AV square is basic, so $v_1 + w_1 = 20$. We know $v_1 = 0$, so $w_1 = 20$.
2. The AW square is basic, so $v_1 + w_2 = 12$. We know $v_1 = 0$, so $w_2 = 12$.
3. The BV square is basic, so $v_2 + w_1 = 15$. We know $w_1 = 20$, so $v_2 = -5$.
4. The BX square is basic, so $v_2 + w_3 = 16$. We know $v_2 = -5$, so $w_3 = 21$.
5. The BY square is basic, so $v_2 + w_4 = 13$. We know $v_2 = -5$, so $w_4 = 18$.
6. The BZ square is basic, so $v_2 + w_5 = 10$. We know $v_2 = -5$, so $w_5 = 15$.
7. The CZ square is basic, so $v_3 + w_5 = 12$. We know $w_5 = 15$, so $v_3 = -3$.

Therefore, the values of the dual variables are as follows:

	V	W	X	Y	Z	
A	[20] **12**	[12] **6**	[15]	[10]	[16]	$v_1 = 0$
B	[15] **2**	[18]	[16] **8**	[13] **10**	[10] **5**	$v_2 = -5$
C	[17]	[14]	[19]	[14]	[12] **12**	$v_3 = -3$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 21$	$w_4 = 18$	$w_5 = 15$	

Next we compute the test values for all of the nonbasic squares. Remember, the test value for a nonbasic square is the cost minus the row variable minus the column variable. We get the following test values:

	V	W	X	Y	Z	
A	[20] **12**	[12] **6**	[15] -6	[10] -8	[16] 1	$v_1 = 0$
B	[15] **2**	[18] 11	[16] **8**	[13] **10**	[10] **5**	$v_2 = -5$
C	[17] 0	[14] 5	[19] 1	[14] -1	[12] **12**	$v_3 = -3$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 21$	$w_4 = 18$	$w_5 = 15$	

We have negative test values, so this solution is not optimal. We choose the most negative test value, which is the -8 in the AY square, as our pivot location. Now we need to find a circuit in this grid that starts in the AY square, goes vertically to a basic square (which must be the BY square, as there are no other basic squares in the Y column), then goes horizontally to another basic square, then goes vertically to another basic square, and so on, until it returns to the AY square. After a little bit of searching, we find the following circuit in the grid, which goes AY–BY–BV–AV–AY:

	V	W	X	Y	Z
A	[20] **12**	[12] **6**	[15] -6	[10] -8	[16] 1
B	[15] **2**	[18] 11	[16] **8**	[13] **10**	[10] **5**
C	[17] 0	[14] 5	[19] 1	[14] -1	[12] **12**

We form a tentative new solution by assigning a flow of t to the AY square, and then alternately subtracting and adding t to the flows in the basic squares around the circuit, leaving the other basic squares unchanged:

	V	W	X	Y	Z
A	$\mathbf{12 - t}$	**6**		$\mathbf{t}$	
B	$\mathbf{2 + t}$		**8**	$\mathbf{10 - t}$	**5**
C					**12**

We want to choose the largest value of t that does not cause any flow to become negative. So we choose $t = 10$ here, which gives us the new basic feasible solution below:

	V	W	X	Y	Z
A	**2**	**6**		**10**	
B	**12**		**8**		**5**
C					**12**

Now we do the whole thing again! Write the costs in the grid, and set $v_1 = 0$:

	V	W	X	Y	Z	
A	20 **2**	12 **6**	15	10 **10**	16	$v_1 = 0$
B	15 **12**	18	16 **8**	13	10 **5**	v_2
C	17	14	19	14	12 **12**	v_3
	w_1	w_2	w_3	w_4	w_5	

Use the basic squares to find the values of the other dual variables:

	V	W	X	Y	Z	
A	20 **2**	12 **6**	15	10 **10**	16	$v_1 = 0$
B	15 **12**	18	16 **8**	13	10 **5**	$v_2 = -5$
C	17	14	19	14	12 **12**	$v_3 = -3$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 21$	$w_4 = 10$	$w_5 = 15$	

Compute the test values for the nonbasic squares:

	V	W	X	Y	Z	
A	20 **2**	12 **6**	15 -6	10 **10**	16 1	$v_1 = 0$
B	15 **12**	18 11	16 **8**	13 8	10 **5**	$v_2 = -5$
C	17 0	14 5	19 1	14 7	12 **12**	$v_3 = -3$
	$w_1 = 20$	$w_2 = 12$	$w_3 = 21$	$w_4 = 10$	$w_5 = 15$	

We have a negative test value (in AX), so this solution is not optimal. The AX square must be the pivot location. Search for a circuit, and find AX–BX–BV–AV–AX:

	V	W	X	Y	Z
A	20 **2**	12 **6**	15 -6	10 **10**	16 1
B	15 **12**	18 11	16 **8**	13 8	10 **5**
C	17 0	14 5	19 1	14 7	12 **12**

Assign a flow of t to the AX square, and then alternately subtract and add t to the flows around the circuit:

	V	W	X	Y	Z
A	$\mathbf{2 - t}$	**6**	$\mathbf{t}$	**10**	
B	$\mathbf{12 + t}$		$\mathbf{8 - t}$		**5**
C					**12**

Choose $t = 2$, which is the largest value that does not produce a negative flow in any square:

	V	W	X	Y	Z
A		**6**	**2**	**10**	
B	**14**		**6**		**5**
C					**12**

Continue! Write the costs in the grid, and set $v_1 = 0$:

	V	W	X	Y	Z	
A	[20]	[12] **6**	[15] **2**	[10] **10**	[16]	$v_1 = 0$
B	[15] **14**	[18]	[16] **6**	[13]	[10] **5**	v_2
C	[17]	[14]	[19]	[14]	[12] **12**	v_3
	w_1	w_2	w_3	w_4	w_5	

Find the values of the other dual variables:

	V	W	X	Y	Z	
A	[20]	[12] **6**	[15] **2**	[10] **10**	[16]	$v_1 = 0$
B	[15] **14**	[18]	[16] **6**	[13]	[10] **5**	$v_2 = 1$
C	[17]	[14]	[19]	[14]	[12] **12**	$v_3 = 3$
	$w_1 = 14$	$w_2 = 12$	$w_3 = 15$	$w_4 = 10$	$w_5 = 9$	

Compute the test values:

	V	W	X	Y	Z	
A	[20] 6	[12] **6**	[15] **2**	[10] **10**	[16] 7	$v_1 = 0$
B	[15] **14**	[18] 5	[16] **6**	[13] 2	[10] **5**	$v_2 = 1$
C	[17] 0	[14] -1	[19] 1	[14] 1	[12] **12**	$v_3 = 3$
	$w_1 = 14$	$w_2 = 12$	$w_3 = 15$	$w_4 = 10$	$w_5 = 9$	

The CW square has a negative test value, so this solution is not optimal. Use the CW square as the pivot location, and find a circuit that starts and ends at CW. This circuit is a little trickier to find than the previous ones, because it isn't just a rectangle: it goes CW–AW–AX–BX–BZ–CZ–CW:

	V	W	X	Y	Z
A	[20] 6	[12] **6**	[15] **2**	[10] **10**	[16] 7
B	[15] **14**	[18] 5	[16] **6**	[13] 2	[10] **5**
C	[17] 0	[14] -1	[19] 1	[14] 1	[12] **12**

Assign a flow of t to the CW square, and then alternately subtract and add t to the flows around the circuit:

	V	W	X	Y	Z
A		$\mathbf{6-t}$	$\mathbf{2+t}$	**10**	
B	**14**		$\mathbf{6-t}$		$\mathbf{5+t}$
C		$\boldsymbol{t}$			$\mathbf{12-t}$

Choose $t = 6$, which is the largest value of t that does not make any flow negative:

	V	W	X	Y	Z
A			**8**	**10**	
B	**14**				**11**
C		**6**			**6**

But now we need to be careful! Look at what happened here. We gained one new basic square (CW), but we lost *two* basic squares (AW and BX), because both of their flows became zero. This means that the number of basic squares has dropped to 6. Remember that the number of basic squares should always be 7 in this problem. To fix this problem, *only one* of the two squares AW and BX whose flow became zero should be made nonbasic; the other one should be kept as a basic square whose flow happens to be zero. (This is a degenerate solution.) Let's choose to keep AW as the basic square. This gives us the following basic feasible solution:

	V	W	X	Y	Z
A		**0**	**8**	**10**	
B	**14**				**11**
C		**6**			**6**

> **Special case when pivoting:** If the flow of more than one basic square becomes zero when a pivot is performed, *only one* of those squares should become nonbasic; the others should be kept as basic squares with flows of zero. This is necessary in order to keep the correct number of basic squares.

Okay, now we can continue. Write the costs in the grid, and set $v_1 = 0$:

	V	W	X	Y	Z	
A	[20]	[12] **0**	[15] **8**	[10] **10**	[16]	$v_1 = 0$
B	[15] **14**	[18]	[16]	[13]	[10] **11**	v_2
C	[17]	[14] **6**	[19]	[14]	[12] **6**	v_3
	w_1	w_2	w_3	w_4	w_5	

Find the values of the other dual variables:

	V	W	X	Y	Z	
A	[20]	[12] **0**	[15] **8**	[10] **10**	[16]	$v_1 = 0$
B	[15] **14**	[18]	[16]	[13]	[10] **11**	$v_2 = 0$
C	[17]	[14] **6**	[19]	[14]	[12] **6**	$v_3 = 2$
	$w_1 = 15$	$w_2 = 12$	$w_3 = 15$	$w_4 = 10$	$w_5 = 10$	

Compute the test values:

	V	W	X	Y	Z	
A	[20] 5	[12] **0**	[15] **8**	[10] **10**	[16] 6	$v_1 = 0$
B	[15] **14**	[18] 6	[16] 1	[13] 3	[10] **11**	$v_2 = 0$
C	[17] 0	[14] **6**	[19] 2	[14] 2	[12] **6**	$v_3 = 2$
	$w_1 = 15$	$w_2 = 12$	$w_3 = 15$	$w_4 = 10$	$w_5 = 10$	

Hurray! All of the test values are nonnegative. That means that this solution is optimal. So an optimal solution to this transportation problem is as follows:

- Ship 8 units from A to X.
- Ship 10 units from A to Y.
- Ship 14 units from B to V.
- Ship 11 units from B to Z.
- Ship 6 units from C to W.
- Ship 6 units from C to Z.

The total cost of this shipment plan is

$$8(\$15) + 10(\$10) + 14(\$15) + 11(\$10) + 6(\$14) + 6(\$12) = \$696,$$

which is the minimum possible cost to satisfy all five demands and exhaust all five supplies.

One other detail deserves mention: the case of a nonunique optimal solution.

Nonuniqueness: If a test value in the optimal grid is zero, then the optimal solution is not unique, and another optimal solution can be obtained by pivoting on that square.

In this case, the test value of 0 in the CV square in the final grid tells us that this optimal solution is not unique—if we like, we can get another optimal solution by pivoting on the CV square. Try it!

A special case when forming the initial basic feasible solution: Occasionally, when you are forming the initial basic feasible solution, you will simultaneously satisfy a demand *and* exhaust a supply (before the bottom right square). When this happens, you need to be careful that you have the right number of basic squares.

Here is an example. Suppose there are four warehouses (A, B, C, and D) with respective supplies of 5, 9, 6, and 4 units, and six customers (U, V, W, X, Y, and Z) with respective demands of 3, 7, 4, 2, 3, and 5 units. When we form the initial basic feasible solution, we get the following:

	U: 3	V: 7	W: 4	X: 2	Y: 3	Z: 5
A: 5	**3**	**2**				
B: 9		**5**	**4**			
C: 6				**2**	**3**	**1**
D: 4						**4**

Notice what happened when we satisfied the demand of W: we simultaneously exhausted the supply of B. This led to a diagonal step in the basic squares, from BW to CX. This disrupts the "stair-step" pattern, and more importantly it gives us too few basic squares. Remember that the number of basic squares should always equal the number of rows plus the number of columns minus 1, which in this case is $4 + 6 - 1 = 9$; but we have only eight basic squares in this solution.

We need to fix this deficiency, and at the same time we need to restore the necessary "stair-step" pattern for the initial basic feasible solution. So we need to add a basic square with a flow of zero. This basic square should be either the CW square or the BX square (but not both!) in order to restore the "stair-step" pattern. Suppose that we choose the CW square; then our initial basic feasible solution will be as follows:

	U: 3	V: 7	W: 4	X: 2	Y: 3	Z: 5
A: 5	**3**	**2**				
B: 9		**5**	**4**			
C: 6			**0**	**2**	**3**	**1**
D: 4						**4**

Special case for the initial basic feasible solution: If a demand is satisfied and a supply is exhausted simultaneously when forming the initial basic feasible solution, then add a basic square with zero flow to restore the "stair-step" pattern and give the right number of basic variables.