Combinatorial Optimization

Problem set 8

Assigned Thursday, July 2, 2015. Due Thursday, July 9, 2015.

1. Fix constants $a \in \mathbb{R}$ and $b > 1$. For $n \in \mathbb{N}$, let $f(n) = n^a$ and $g(n) = b^n$. Prove that $f(n) = o\big(g(n)\big)$.

2. Carefully state the decision (recognition) version of the minimum spanning tree problem. Prove that this problem is in P.

3. Prove that if we had a polynomial-time algorithm for *computing the length* of the shortest TSP tour, then we would have a polynomial-time algorithm for *finding* the shortest TSP tour. (In other words, show how to solve the optimization version of the traveling salesman problem in polynomial time given a polynomial-time algorithm for solving the evaluation version.)

[Hint: It may be helpful first to assume that the optimal tour is unique, and then later revise your approach to remove that assumption. If the optimal tour is unique, then, of course, each arc either is in the optimal tour you're searching for or else is not in any optimal tour.]

4. In the VERTEX COVER problem, an instance is a simple undirected graph $G = (V, E)$ and a positive integer $k \leq |V|$, and the question is whether there exists a *vertex cover* of size no greater than k, that is, a subset $V' \subseteq V$ with $|V'| \leq k$ such that for every edge $\{u, v\} \in E$ at least one of the endpoints u and v belongs to V'. Prove that VERTEX COVER is in NP.

[To do this, you will need to carefully describe the form of a certificate and the operation of a corresponding verifier (i.e., a certificate-checking algorithm) and show that every "yes" instance of the VERTEX COVER problem has a polynomial-size certificate that can be verified in polynomial time by the verifier.]

5. In the 3-COLORABILITY problem, an instance is a simple undirected graph $G = (V, E)$, and the question is whether there exists a proper 3-coloring of the vertices of G, that is, a function $f : V \to \{1, 2, 3\}$ such that $f(u) \neq f(v)$ for every edge $\{u, v\} \in E$. (Think of the function f as assigning each vertex a "color" in the set $\{1, 2, 3\}$; then no two adjacent vertices can be assigned the same color.) The 4-COLORABILITY problem is defined analogously. Describe and justify a polynomial-time transformation from 3-COLORABILITY to 4-COLORABILITY.

6. In the SUBGRAPH ISOMORPHISM problem, an instance consists of two simple undirected graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, and the question is whether G contains a subgraph isomorphic to H, that is, a subgraph $K = (V_K, E_K)$ with $V_K \subseteq V_G$ and $E_K \subseteq E_G$ such that there exists a bijection $f : V_H \to V_K$ with $\{u, v\} \in E_H$ if and only if $\{f(u), f(v)\} \in E_K$. Prove that SUBGRAPH ISOMORPHISM is NP-complete.