Combinatorial Optimization

Problem set 5

Assigned Monday, June 15, 2015. Due Thursday, June 18, 2015.

1. Consider the problem of determining the least expensive way to complete a project by a given deadline (as we studied last week). When the linear program is formulated, the objective function has a constant term. For instance, if activities A, B, and C have usual times of 8, 5, and 7 days and can be sped up at a cost of \$200, \$150, and \$225 per day, respectively, then the objective function (i.e., the total speedup cost) is

$$200(8 - d_{\mathrm{A}}) + 150(5 - d_{\mathrm{B}}) + 225(7 - d_{\mathrm{C}}),$$

where d_{A}, d_{B}, and d_{C} are duration variables for the three activities. When expanded, this objective function becomes

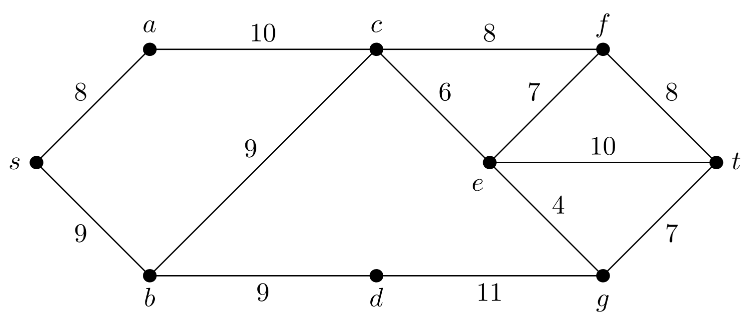
$$3925 - 200d_{\mathrm{A}} - 150d_{\mathrm{B}} - 225d_{\mathrm{C}},$$

which has the constant term 3925. Describe a way to handle an objective function with a constant term in the simplex algorithm.

2. What is the greatest possible number of critical paths in a project with n activities? Describe a family of examples for infinitely many values of n that attain this number of critical paths.

3. Consider a directed graph $G = (V, E)$ with nonnegative edge weights $c_{ij} \geq 0$ and specified nodes $s, t \in V$. For each node $i \in V$, let π_i be the distance of a shortest (directed) path from i to t. (Assume that every node i has such a path to t.) Show that π is an optimal feasible solution to the dual of the node-arc LP formulation for the shortest path problem on G from s to t. Is the assumption $c_{ij} \geq 0$ necessary?

4. Use Dijkstra's algorithm (or the primal-dual algorithm) to find a shortest path from s to t in the following undirected graph. [The first question to consider is how to use one of these algorithms to find a shortest path in an *undirected* graph.]



5. Give an example of a simple graph with at least two vertices such that no two vertices have the same degree, or explain why this is impossible.

6. Let $G = (V, E)$ be a simple undirected graph, and let $n = |V|$. Prove that all of the following statements are equivalent.

(a) G is a tree (that is, G is connected and acyclic).

(b) For any two distinct vertices $u, v \in V$, there exists a unique path in G between u and v.

(c) G is minimally connected: G is connected, but if any edge is removed from G then the resulting graph is disconnected.

(d) G is maximally acyclic: G is acyclic, but if any edge is added joining nonadjacent vertices of G then the resulting graph has a cycle.

(e) G is connected and has $n - 1$ edges.

(f) G is acyclic and has $n - 1$ edges.