Combinatorial Optimization

Problem set 4: solutions

1. Consider a project consisting of the following nine activities.

Activity	Immediate prerequisites	Usual time (days)	Crash time (days)	Cost per day to speed up
A	—	2	—	—
B	—	6	3	\$180
C	A	4	2	\$150
D	B	2	1	\$200
E	B	4	1	\$ 75
F	C, D	3	1	\$250
G	D	1	—	—
H	F	3	2	\$100
I	E, F, G	4	1	\$140

(a) Draw a CPM network for this project.

(b) Using the *usual times*:

(i) Determine the earliest and latest times for each node.

(ii) Determine the float for each activity.

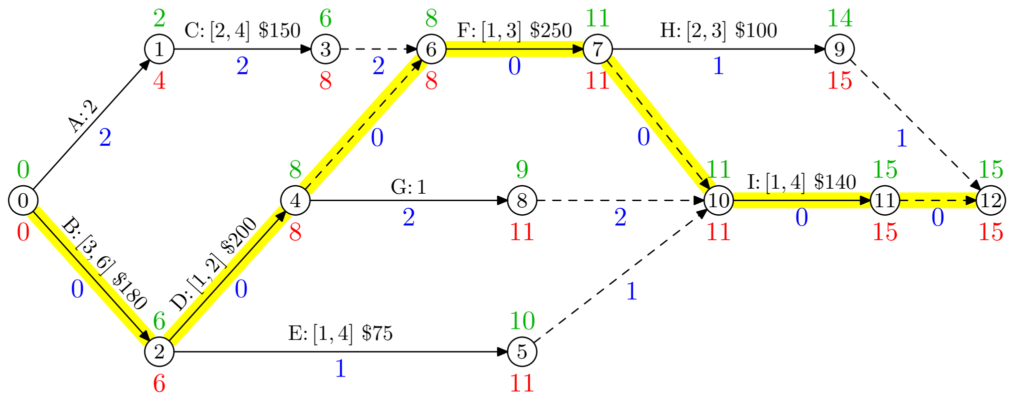
(iii) Determine the critical path.

(c) Formulate a linear program to determine the least expensive way to reduce the length of the project by 4 days. Solve your linear program (with Maple or otherwise) and interpret your results.

(d) Formulate a linear program to determine the shortest possible completion time that can be achieved with a budget of \$900. Solve your linear program and interpret your results.

▷ **Solution.**

(a) A CPM network for this project is shown below. Solid arrows represent activities. The edge label for an activity that cannot be sped up gives the duration of the activity; the label for an activity that can be sped up gives the interval of possible durations (from the crash time to the usual time) and the per-day speedup cost. This drawing also incorporates the information from part (b): the earliest time for each node (using the usual times) is shown in green above the node, the latest time for each node is shown in red below the node, the float for each activity is shown in blue below the corresponding arrow, and the critical path is highlighted in yellow.



(b) Let u_i denote the usual time of activity i. Let e_j and l_j denote the earliest time and latest time of node j, respectively, using the usual times.

(i) The earliest time of node 0 is 0. For $j \neq 0$ the earliest time of node j is found as follows: for each incoming edge (including dummy edges), add the earliest time of the edge's start node to the duration (i.e., the usual time) of the corresponding activity; and then take the maximum of all of these sums. Note that the duration of a dummy edge is zero.

Node	Earliest time
0	0
1	$e_0 + u_{\mathrm{A}} = 0 + 2 = 2$
2	$e_0 + u_{\mathrm{B}} = 0 + 6 = 6$
3	$e_1 + u_{\mathrm{C}} = 2 + 4 = 6$
4	$e_2 + u_{\mathrm{D}} = 6 + 2 = 8$
5	$e_2 + u_{\mathrm{E}} = 6 + 4 = 10$
6	$\max\{e_3 + 0, e_4 + 0\} = \max\{6, 8\} = 8$
7	$e_6 + u_{\mathrm{F}} = 8 + 3 = 11$
8	$e_4 + u_{\mathrm{G}} = 8 + 1 = 9$
9	$e_7 + u_{\mathrm{H}} = 11 + 3 = 14$
10	$\max\{e_5 + 0, e_7 + 0, e_8 + 0\} = \max\{10, 11, 9\} = 11$
11	$e_{10} + u_{\mathrm{I}} = 11 + 4 = 15$
12	$\max\{e_9 + 0, e_{11} + 0\} = \max\{14, 15\} = 15$

The latest time of the last node is equal to its earliest time. For each other node, the latest time is found as follows: for each outgoing edge (including dummy edges), subtract the duration of the corresponding activity from the latest time of the edge's end node; and then take the minimum of all of these differences.

Node	Latest time
12	$e_{12} = 15$
11	$l_{12} - 0 = 15$
10	$l_{11} - u_{\mathrm{I}} = 15 - 4 = 11$
9	$l_{12} - 0 = 15$
8	$l_{10} - 0 = 11$
7	$\min\{l_9 - u_{\mathrm{H}}, l_{10} - 0\} = \min\{15 - 3, 11\} = 11$
6	$l_{11} - u_{\mathrm{F}} = 11 - 3 = 8$
5	$l_{10} - 0 = 11$
4	$\min\{l_6 - 0, l_8 - u_{\mathrm{G}}\} = \min\{8, 11 - 1\} = 8$
3	$l_6 - 0 = 8$
2	$\min\{l_4 - u_{\mathrm{D}}, l_5 - u_{\mathrm{E}}\} = \min\{8 - 2, 11 - 4\} = 6$
1	$l_3 - u_{\mathrm{C}} = 8 - 4 = 4$
0	$\min\{l_1 - u_{\mathrm{A}}, l_2 - u_{\mathrm{B}}\} = \min\{4 - 2, 6 - 6\} = 0$

(ii) The float of an activity is equal to the latest time of its end node, minus the earliest time of its start node, minus its duration.

Activity	Float
A	$l_1 - e_0 - u_{\mathrm{A}} = 4 - 0 - 2 = 2$
B	$l_2 - e_0 - u_{\mathrm{B}} = 6 - 0 - 6 = 0$
C	$l_3 - e_1 - u_{\mathrm{C}} = 8 - 2 - 4 = 2$
D	$l_4 - e_2 - u_{\mathrm{D}} = 8 - 6 - 2 = 0$
E	$l_5 - e_2 - u_{\mathrm{E}} = 11 - 6 - 4 = 1$
F	$l_7 - e_6 - u_{\mathrm{F}} = 11 - 8 - 3 = 0$
G	$l_8 - e_4 - u_{\mathrm{G}} = 11 - 8 - 1 = 2$
H	$l_9 - e_7 - u_{\mathrm{H}} = 15 - 11 - 3 = 1$
I	$l_{11} - e_{10} - u_{\mathrm{I}} = 15 - 11 - 4 = 0$

For completeness, we can also compute the floats for the dummy activities, even though they do not represent actual activities in the project:

Dummy activity	Float
$3 \to 6$	$l_6 - e_3 - 0 = 8 - 6 - 0 = 2$
$4 \to 6$	$l_6 - e_4 - 0 = 8 - 8 - 0 = 0$
$5 \to 10$	$l_{10} - e_5 - 0 = 11 - 10 - 0 = 1$
$7 \to 10$	$l_{10} - e_7 - 0 = 11 - 11 - 0 = 0$
$8 \to 10$	$l_{10} - e_8 - 0 = 11 - 9 - 0 = 2$
$9 \to 12$	$l_{12} - e_9 - 0 = 15 - 14 - 0 = 1$
$11 \to 12$	$l_{12} - e_{11} - 0 = 15 - 15 - 0 = 0$

(iii) A critical path is a path from the beginning of the project (node 0) to the completion of the project (node 12) consisting entirely of critical activities. There is a unique critical path in this project, consisting of the activities B—D—F—I.

(c, d) First we define the variables and their domains. For $i \in \{\mathrm{B, C, D, E, F, H, I}\}$, let d_i denote the duration of activity i (possibly sped up from its usual time). For $0 \le j \le 12$, let t_j denote the time at which node j occurs. All of these variables are nonnegative.

Next we list the network constraints, because we will need the network constrains in any linear program for this project, and these constraints come from the CPM network we drew in part (a). Each activity gives us a sequence constraint:

$$\begin{array}{lll} t_1 - t_0 \ge 2 & t_4 - t_2 \ge d_{\mathrm{D}} & t_8 - t_4 \ge 1 \\ t_2 - t_0 \ge d_{\mathrm{B}} & t_5 - t_2 \ge d_{\mathrm{E}} & t_9 - t_7 \ge d_{\mathrm{H}} \\ t_3 - t_1 \ge d_{\mathrm{C}} & t_7 - t_6 \ge d_{\mathrm{F}} & t_{11} - t_{10} \ge d_{\mathrm{I}} \end{array}$$

Each dummy edge gives us a dummy constraint:

$$\begin{array}{lll} t_6 - t_3 \ge 0 & t_{10} - t_5 \ge 0 & t_{12} - t_9 \ge 0 \\ t_6 - t_4 \ge 0 & t_{10} - t_7 \ge 0 & t_{12} - t_{11} \ge 0 \\ & t_{10} - t_8 \ge 0 & \end{array}$$

Each activity that can be sped up gives us a pair of duration constraints:

$$\begin{array}{lll} 3 \le d_{\mathrm{B}} \le 6 & 1 \le d_{\mathrm{E}} \le 4 & 2 \le d_{\mathrm{H}} \le 3 \\ 2 \le d_{\mathrm{C}} \le 4 & 1 \le d_{\mathrm{F}} \le 3 & 1 \le d_{\mathrm{I}} \le 4 \\ 1 \le d_{\mathrm{D}} \le 2 & & \end{array}$$

Now, our objective in part (c) is to minimize the total speedup cost, which is

$$180(6-d_{\mathrm{B}})+150(4-d_{\mathrm{C}})+200(2-d_{\mathrm{D}})+75(4-d_{\mathrm{E}})+250(3-d_{\mathrm{F}})+100(3-d_{\mathrm{H}})+140(4-d_{\mathrm{I}}).$$

The usual project completion time is 15 days, so reducing the length of the project by 4 days means completing it in 11 days, which gives us the deadline constraint $t_{12} \le 11$. Therefore our linear program for part (c) is

$$\begin{array}{rl} \text{minimize} & \text{[total speedup cost]} \\ \text{subject to} & \text{[all network constraints]} \\ & t_{12} \le 11 \\ & \text{all variables nonnegative.} \end{array}$$

Our objective in part (d) is to minimize the project completion time, which is t_{12}, subject to the budget constraint [total speedup cost] ≤ 900. So our linear program for part (d) is

$$\begin{aligned} \text{minimize} \quad & t_{12} \\ \text{subject to} \quad & \text{[all network constraints]} \\ & \text{[total speedup cost]} \leq 900 \\ & \text{all variables nonnegative.} \end{aligned}$$

The following Maple worksheet solves both of these linear programs. (It is convenient to solve them together because they share almost all of their constraints.)

```
> restart;
> with(Optimization);
```

$[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]$

```
> sequence:=[t1-t0>=2,t2-t0>=dB,t3-t1>=dC,t4-t2>=dD,t5-t2>=dE,
  t7-t6>=dF,t8-t4>=1,t9-t7>=dH,t11-t10>=dI];
```

$sequence := [2 \leq t1 - t0, dB \leq t2 - t0, dC \leq t3 - t1, dD \leq t4 - t2, dE \leq t5 - t2, dF \leq t7 - t6, 1 \leq t8 - t4, dH \leq t9 - t7, dI \leq t11 - t10]$

```
> dummy:=[t6-t3>=0,t6-t4>=0,t10-t5>=0,t10-t7>=0,t10-t8>=0,
  t12-t9>=0,t12-t11>=0];
```

$dummy := [0 \leq t6 - t3, 0 \leq t6 - t4, 0 \leq t10 - t5, 0 \leq t10 - t7, 0 \leq t10 - t8, 0 \leq t12 - t9, 0 \leq t12 - t11]$

```
> duration:=[3<=dB,dB<=6,2<=dC,dC<=4,1<=dD,dD<=2,1<=dE,dE<=4,
  1<=dF,dF<=3,2<=dH,dH<=3,1<=dI,dI<=4];
```

$duration := [3 \leq dB, dB \leq 6, 2 \leq dC, dC \leq 4, 1 \leq dD, dD \leq 2, 1 \leq dE, dE \leq 4, 1 \leq dF, dF \leq 3, 2 \leq dH, dH \leq 3, 1 \leq dI, dI \leq 4]$

```
> network:=[op(sequence),op(dummy),op(duration)];
```

$network := [2 \leq t1-t0, dB \leq t2-t0, dC \leq t3-t1, dD \leq t4-t2, dE \leq t5-t2, dF \leq t7 - t6, 1 \leq t8 - t4, dH \leq t9 - t7, dI \leq t11 - t10, 0 \leq t6 - t3, 0 \leq t6 - t4, 0 \leq t10 - t5, 0 \leq t10 - t7, 0 \leq t10 - t8, 0 \leq t12 - t9, 0 \leq t12 - t11, 3 \leq dB, dB \leq 6, 2 \leq dC, dC \leq 4, 1 \leq dD, dD \leq 2, 1 \leq dE, dE \leq 4, 1 \leq dF, dF \leq 3, 2 \leq dH, dH \leq 3, 1 \leq dI, dI \leq 4]$

```
> cost:=(dB,dC,dD,dE,dF,dH,dI)->180*(6-dB)+150*(4-dC)
  +200*(2-dD)+75*(4-dE)+250*(3-dF)+100*(3-dH)+140*(4-dI);
```

$cost := (dB, dC, dD, dE, dF, dH, dI) \rightarrow 3990 - 180\, dB - 150\, dC - 200\, dD - 75\, dE - 250\, dF - 100\, dH - 140\, dI$

```
> constraintsC:=[op(network),t12<=11];
```

$constraintsC := [2 \leq t1 - t0, dB \leq t2 - t0, dC \leq t3 - t1, dD \leq t4 - t2, dE \leq t5 - t2, dF \leq t7 - t6, 1 \leq t8 - t4, dH \leq t9 - t7, dI \leq t11 - t10, 0 \leq t6 - t3, 0 \leq t6 - t4, 0 \leq t10 - t5, 0 \leq t10 - t7, 0 \leq t10 - t8, 0 \leq t12 - t9, 0 \leq t12 - t11, 3 \leq dB, dB \leq 6, 2 \leq dC, dC \leq 4, 1 \leq dD, dD \leq 2, 1 \leq dE, dE \leq 4, 1 \leq dF, dF \leq 3, 2 \leq dH, dH \leq 3, 1 \leq dI, dI \leq 4, t12 \leq 11]$

```
> LPSolve(cost(dB,dC,dD,dE,dF,dH,dI),constraintsC,
    assume=nonnegative);
```

$$\big[739.99999883316, [dB = 4.00000000068840, dC = 4., dD = 2.00000000034420, dE = 4., dF = 3., dH = 2.00000000275359, dI = 2.00000000499089, t0 = 0., t1 = 2., t10 = 8.99999999535331, t11 = 11.0000000003442, t12 = 11., t2 = 3.99999999965580, t3 = 5.99999999896740, t4 = 5.99999999896740, t5 = 8.99999999569751, t6 = 5.99999999862320, t7 = 8.99999999759060, t8 = 6.99999999896740, t9 = 11.]\big]$$

```
> constraintsD:=[op(network),cost(dB,dC,dD,dE,dF,dH,dI)<=900];
```

$$constraintsD := [2 \le t1 - t0, dB \le t2 - t0, dC \le t3 - t1, dD \le t4 - t2, dE \le t5 - t2, dF \le t7 - t6, 1 \le t8 - t4, dH \le t9 - t7, dI \le t11 - t10, 0 \le t6 - t3, 0 \le t6 - t4, 0 \le t10 - t5, 0 \le t10 - t7, 0 \le t10 - t8, 0 \le t12 - t9, 0 \le t12 - t11, 3 \le dB, dB \le 6, 2 \le dC, dC \le 4, 1 \le dD, dD \le 2, 1 \le dE, dE \le 4, 1 \le dF, dF \le 3, 2 \le dH, dH \le 3, 1 \le dI, dI \le 4, -180dB - 150dC - 200dD - 75dE - 250dF - 100dH - 140dI \le -3090]$$

```
> LPSolve(t12,constraintsD,assume=nonnegative);
```

$$\big[10.3599999956080, [dB = 4.00000000034420, dC = 4., dD = 2.00000000000000, dE = 4., dF = 2.35999999801741, dH = 2., dI = 2.00000000309779, t0 = 0., t1 = 2., t10 = 8.35999999560802, t11 = 10.3599999956080, t12 = 10.3599999956080, t2 = 4., t3 = 5.99999999965580, t4 = 5.99999999965580, t5 = 7.99999999965580, t6 = 5.99999999965580, t7 = 8.35999999664062, t8 = 6.99999999965580, t9 = 10.3599999966406]\big]$$

The optimal activity durations in part (c) are $d_{\mathrm{B}} = 4$, $d_{\mathrm{C}} = 4$, $d_{\mathrm{D}} = 2$, $d_{\mathrm{E}} = 4$, $d_{\mathrm{F}} = 3$, $d_{\mathrm{H}} = 2$, and $d_{\mathrm{I}} = 2$. This means that the least expensive way to reduce the length of the project by 4 days is to speed up activity B by 2 days from its usual time, activity H by 1 day from its usual time, and activity I by 2 days from its usual time. The optimal objective value (i.e., the minimum total speedup cost) is $740.

The optimal activity durations in part (d) are $d_{\mathrm{B}} = 4$, $d_{\mathrm{C}} = 4$, $d_{\mathrm{D}} = 2$, $d_{\mathrm{E}} = 4$, $d_{\mathrm{F}} = 2.36$, $d_{\mathrm{H}} = 2$, and $d_{\mathrm{I}} = 2$, with an optimal objective value of 10.36 days. This means that the best way to apply the budget of $900 in order to reduce the project completion time is to speed up activity B by 2 days from its usual time, activity F by 0.64 day from its usual time, activity H by 1 day from its usual time, and activity I by 2 days from its usual time. □

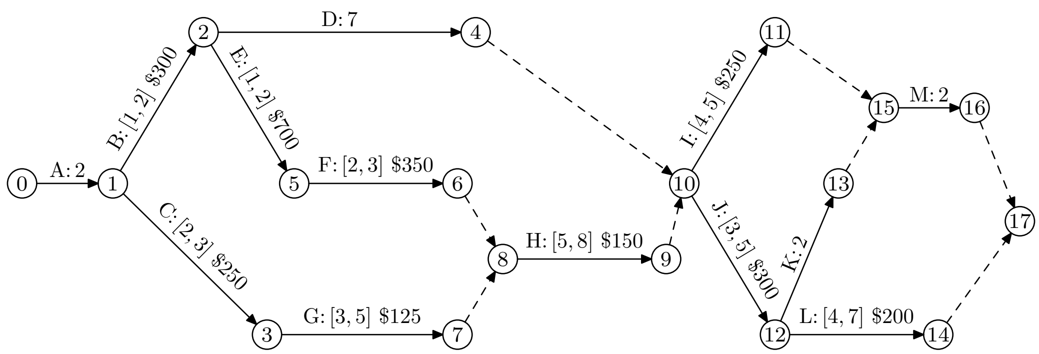
2. Rural Residence, Inc. (RRI) manufactures and builds prefab log homes. The logs are cut at their plant and delivered to the site. All other materials, such as roofing, doors, windows, etc., are purchased from other companies. The tasks involved in building one of their homes are shown in the table below.

Activity	Immediate prerequisites	Usual time (days)	Crash time (days)	Cost per day to speed up
A. Prepare site	—	2	—	—
B. Adjust design to site	A	2	1	\$300
C. Cut logs for house	A	3	2	\$250
D. Obtain other materials	B	7	—	—
E. Excavate basement	B	2	1	\$700
F. Pour foundation	E	3	2	\$350
G. Ship logs	C	5	3	\$125
H. Assemble logs	F, G	8	5	\$150
I. Complete roof, doors, etc.	D, H	5	4	\$250
J. Prepare for utilities	D, H	5	3	\$300
K. Connect utilities	J	2	—	—
L. Finish interior	J	7	4	\$200
M. Landscape lot	I, K	2	—	—

In addition to the tasks above, RRI must maintain a trailer and security guard at the job site from the time that the excavation of the basement begins until the interior is finished, at a cost of \$210 per day.

Determine a project schedule that will minimize the total cost of the project.

▷ **Solution.** A CPM network for this project is shown below. Note that we can reuse node 10 as the start node for both activities I and J, because they have the same prerequisites. The excavation of the basement (activity E) begins at node 2, and the finishing of the interior (activity L) ends at node 14, so the trailer and security guard must be maintained between the two times represented by those nodes.



We will formulate a linear program for this problem, based on the CPM network. For $i \in \{\mathrm{B,C,E,F,G,H,I,J,L}\}$, let d_i denote the duration of activity i (possibly sped up from its usual time). For $0 \le j \le 17$, let t_j denote the time at which node j occurs. All of these variables are nonnegative.

The network constraints come from the CPM network. Each activity gives us a sequence constraint:

$$
\begin{aligned}
&t_1 - t_0 \ge 2 && t_6 - t_5 \ge d_{\mathrm{F}} && t_{12} - t_{10} \ge d_{\mathrm{J}}\\
&t_2 - t_1 \ge d_{\mathrm{B}} && t_7 - t_3 \ge d_{\mathrm{G}} && t_{13} - t_{12} \ge 2\\
&t_3 - t_1 \ge d_{\mathrm{C}} && t_9 - t_8 \ge d_{\mathrm{H}} && t_{14} - t_{12} \ge d_{\mathrm{L}}\\
&t_4 - t_2 \ge 7 && t_{11} - t_{10} \ge d_{\mathrm{I}} && t_{16} - t_{15} \ge 2\\
&t_5 - t_2 \ge d_{\mathrm{E}} && &&
\end{aligned}
$$

Each dummy edge gives us a dummy constraint:

$$t_8 - t_6 \geq 0 \qquad t_{10} - t_9 \geq 0 \qquad t_{17} - t_{14} \geq 0$$
$$t_8 - t_7 \geq 0 \qquad t_{15} - t_{11} \geq 0 \qquad t_{17} - t_{16} \geq 0$$
$$t_{10} - t_4 \geq 0 \qquad t_{15} - t_{13} \geq 0$$

Each activity that can be sped up gives us a pair of duration constraints:

$$1 \leq d_{\mathrm{B}} \leq 2 \qquad 2 \leq d_{\mathrm{F}} \leq 3 \qquad 4 \leq d_{\mathrm{I}} \leq 5$$
$$2 \leq d_{\mathrm{C}} \leq 3 \qquad 3 \leq d_{\mathrm{G}} \leq 5 \qquad 3 \leq d_{\mathrm{J}} \leq 5$$
$$1 \leq d_{\mathrm{E}} \leq 2 \qquad 5 \leq d_{\mathrm{H}} \leq 8 \qquad 4 \leq d_{\mathrm{L}} \leq 7$$

Our objective is to minimize the total project cost. The portion of the project cost under our control is the speedup cost, plus the cost of maintaining the trailer and security guard from the time of node 2 until the time of node 14 at \$210 per day. Therefore, the quantity we aim to minimize is

$$300(2-d_{\mathrm{B}}) + 250(3-d_{\mathrm{C}}) + 700(2-d_{\mathrm{E}}) + 350(3-d_{\mathrm{F}}) + 125(5-d_{\mathrm{G}})$$
$$+ 150(8-d_{\mathrm{H}}) + 250(5-d_{\mathrm{I}}) + 300(5-d_{\mathrm{J}}) + 200(7-d_{\mathrm{L}}) + 210(t_{14}-t_2).$$

Our linear program is

$$\begin{array}{rl} \text{minimize} & \text{[cost]} \\ \text{subject to} & \text{[all network constraints]} \\ & \text{all variables nonnegative.} \end{array}$$

The following Maple worksheet solves this linear program.

```
> restart;
> with(Optimization);
```

$[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]$

```
> sequence:=[t1-t0>=2,t2-t1>=dB,t3-t1>=dC,t4-t2>=7,t5-t2>=dE,
   t6-t5>=dF,t7-t3>=dG,t9-t8>=dH,t11-t10>=dI,t12-t10>=dJ,
   t13-t12>=2,t14-t12>=dL,t16-t15>=2];
```

$sequence := [2 \leq t1 - t0, dB \leq t2 - t1, dC \leq t3 - t1, 7 \leq t4 - t2, dE \leq t5 - t2, dF \leq t6 - t5, dG \leq t7 - t3, dH \leq t9 - t8, dI \leq t11, t10, dJ \leq t12 - t10, 2 \leq t13 - t12, dL \leq t14 - t12, 2 \leq t16 - t15]$

```
> dummy:=[t8-t6>=0,t8-t7>=0,t10-t4>=0,t10-t9>=0,t15-t11>=0,
   t15-t13>=0,t17-t14>=0,t17-t16>=0];
```

$dummy := [0 \leq t8 - t6, 0 \leq t8 - t7, 0 \leq t10 - t4, 0 \leq t10 - t9, 0 \leq t15 - t11, 0 \leq t15 - t13, 0 \leq t17 - t14, 0 \leq t17 - t16]$

```
> duration:=[1<=dB,dB<=2,2<=dC,dC<=3,1<=dE,dE<=2,2<=dF,dF<=3,3<=dG,
   dG<=5,5<=dH,dH<=8,4<=dI,dI<=5,3<=dJ,dJ<=5,4<=dL,dL<=7];
```

$duration := [1 \leq dB, dB \leq 2, 2 \leq dC, dC \leq 3, 1 \leq dE, dE \leq 2, 2 \leq dF, dF \leq 3, 3 \leq dG, dG \leq 5, 5 \leq dH, dH \leq 8, 4 \leq dI, dI \leq 5, 3 \leq dJ, dJ \leq 5, 4 \leq dL, dL \leq 7]$

```
> network:=[op(sequence),op(dummy),op(duration)];
```

$$network := [2 \le t1 - t0, dB \le t2 - t1, dC \le t3 - t1, 7 \le t4 - t2, dE \le t5 - t2, dF \le t6 - t5, dG \le t7 - t3, dH \le t9 - t8, dI \le t11, t10, dJ \le t12 - t10, 2 \le t13 - t12, dL \le t14 - t12, 2 \le t16 - t15, 0 \le t8 - t6, 0 \le t8 - t7, 0 \le t10 - t4, 0 \le t10 - t9, 0 \le t15 - t11, 0 \le t15 - t13, 0 \le t17 - t14, 0 \le t17 - t16, 1 \le dB, dB \le 2, 2 \le dC, dC \le 3, 1 \le dE, dE \le 2, 2 \le dF, dF \le 3, 3 \le dG, dG \le 5, 5 \le dH, dH \le 8, 4 \le dI, dI \le 5, 3 \le dJ, dJ \le 5, 4 \le dL, dL \le 7]$$

```
> cost:=(dB,dC,dE,dF,dG,dH,dI,dJ,dL,t2,t14)->300*(2-dB)+250*(3-dC)
  +700*(2-dE)+350*(3-dF)+125*(5-dG)+150*(8-dH)+250*(5-dI)
  +300*(5-dJ)+200*(7-dL)+210*(t14-t2);
```

$$cost := (dB, dC, dE, dF, dG, dH, dI, dJ, dL, t2, t14) \to 9775 - 300\,dB - 250\,dC - 700\,dE - 350\,dF - 125\,dG - 150\,dH - 250\,dI - 300\,dJ - 200\,dL + 210\,t14 - 210\,t2$$

```
> LPSolve(cost(dB,dC,dE,dF,dG,dH,dI,dJ,dL,t2,t14),network,
  assume=nonnegative);
```

$$[5039.99999877809, [dB = 2.00000000137680, dC = 3.00000000034420, dE = 2., dF = 3., dG = 5., dH = 5., dI = 5., dJ = 5., dL = 4., t0 = 0., t1 = 2., t10 = 14.9999999977627, t11 = 21.9999999974185, t12 = 19.9999999974185, t13 = 21.9999999974185, t14 = 23.9999999970743, t15 = 21.9999999970743, t16 = 23.9999999970743, t17 = 23.9999999967301, t2 = 5.00000000051630, t3 = 4.99999999931160, t4 = 14.9999999981069, t5 = 6.99999999948370, t6 = 9.99999999845110, t7 = 9.99999999827900, t8 = 9.99999999810690, t9 = 14.9999999977627]]$$

The optimal activity durations are $d_B = 2$, $d_C = 3$, $d_E = 2$, $d_F = 3$, $d_G = 5$, $d_H = 5$, $d_I = 5$, $d_J = 5$, and $d_L = 4$. This means that the way to minimize the total cost of the project is to speed up each of activities H and L by 3 days from their usual times. The optimal objective value (i.e., the minimum total speedup cost and maintenance cost for the trailer and security guard) is \$5040. The project will be completed in 24 days under this schedule. □

3. The ABC Co. manufactures its product in two plants, A and B, and sells its product in four markets, W, X, Y, and Z. The capacity in Plant A is 300 units, and in Plant B it is 350 units. The demands and per-unit shipping costs for the four markets are shown below.

	Markets			
	W	X	Y	Z
Demand:	155	230	225	160
Per-unit shipping cost from A:	\$10	\$20	\$15	\$25
Per-unit shipping cost from B:	\$ 5	\$15	\$10	\$20

The usual per-unit labor cost is \$95 in either plant. The other costs per unit are \$50 in Plant A and \$70 in Plant B. Overtime labor can be hired only at Plant A at a per-unit cost of \$140. If the capacity is not adequate to meet demand, additional items can be manufactured at Plant A using overtime labor.

(a) Formulate a linear program to determine how ABC should schedule its production to meet all demand while minimizing its total costs. Solve your LP (with Maple or otherwise) and interpret the results.

(b) Solve this problem using the specialized transportation algorithm described in class. Compare this result to your result from part (a).

▷ **Solution.** We can view this as an instance of the transportation problem. We have three origins: usual production in Plant A, usual production in Plant B, and overtime production in Plant A. We shall label these origins A, B, and O, respectively. We have four destinations: the markets W, X, Y, and Z.

The total demand is $155 + 230 + 225 + 160 = 770$ units, but the total supply at origins A and B (that is, the total capacity of both plants) is only 650 units. Therefore, we will need to produce 120 units using overtime labor, so the supply at origin O is 120.

The total per-unit production costs at each origin are \$145 at A, \$165 at B, and \$190 at O. Combining these with the shipping costs, we get the per-unit costs from each source to each destination shown in the following table.

	W	X	Y	Z
A	\$155	\$165	\$160	\$170
B	\$170	\$180	\$175	\$185
O	\$200	\$210	\$205	\$215

(a) For $i \in \{\mathrm{A}, \mathrm{B}, \mathrm{O}\}$ and $j \in \{\mathrm{W}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$, let x_{ij} denote the number of units to be produced at origin i and shipped to market j. All of these variables are nonnegative.

Our objective is to minimize total cost, subject to the constraints that the supply at each origin is exhausted and the demand at each destination is satisfied. Thus we have the following linear program.

$$
\begin{array}{ll}
\text{maximize} & 155x_{\mathrm{AW}} + 165x_{\mathrm{AX}} + 160x_{\mathrm{AY}} + 170x_{\mathrm{AZ}} \\
& + 170x_{\mathrm{BW}} + 180x_{\mathrm{BX}} + 175x_{\mathrm{BY}} + 185x_{\mathrm{BZ}} \\
& + 200x_{\mathrm{OW}} + 210x_{\mathrm{OX}} + 205x_{\mathrm{OY}} + 215x_{\mathrm{OZ}} \\
\text{subject to} & x_{\mathrm{AW}} + x_{\mathrm{AX}} + x_{\mathrm{AY}} + x_{\mathrm{AZ}} = 300 \\
& x_{\mathrm{BW}} + x_{\mathrm{BX}} + x_{\mathrm{BY}} + x_{\mathrm{BZ}} = 350 \\
& x_{\mathrm{OW}} + x_{\mathrm{OX}} + x_{\mathrm{OY}} + x_{\mathrm{OZ}} = 120 \\
& x_{\mathrm{AW}} + x_{\mathrm{BW}} + x_{\mathrm{OW}} = 155 \\
& x_{\mathrm{AX}} + x_{\mathrm{BX}} + x_{\mathrm{OX}} = 230 \\
& x_{\mathrm{AY}} + x_{\mathrm{BY}} + x_{\mathrm{OY}} = 225 \\
& x_{\mathrm{AZ}} + x_{\mathrm{BZ}} + x_{\mathrm{OZ}} = 160 \\
& \text{all variables nonnegative.}
\end{array}
$$

The following Maple worksheet solves this linear program.

```
> restart;
> with(Optimization);
```

$[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]$

```
> cost:=(AW,AX,AY,AZ,BW,BX,BY,BZ,OW,OX,OY,OZ)->155*AW+165*AX
    +160*AY+170*AZ+170*BW+180*BX+175*BY+185*BZ+200*OW+210*OX
    +205*OY+215*OZ;
```

$cost := (AW, AX, AY, AZ, BW, BX, BY, BZ, OW, OX, OY, OZ) \to 155\,AW + 165\,AX + 160\,AY + 170\,AZ + 170\,BW + 180\,BX + 175\,BY + 185\,BZ + 200\,OW + 210\,OX + 205\,OY + 215\,OZ$

```
> supply:=[AW+AX+AY+AZ=300,BW+BX+BY+BZ=350,OW+OX+OY+OZ=120];
```

$$\mathit{supply} := [AW + AX + AY + AZ = 300, BW + BX + BY + BZ = 350,$$
$$OW + OX + OY + OZ = 120]$$

```
> demand:=[AW+BW+OW=155,AX+BX+OX=230,AY+BY+OY=225,AZ+BZ+OZ=160];
```

$$\mathit{demand} := [AW + BW + OW = 155, AX + BX + OX = 230,$$
$$AY + BY + OY = 225, AZ + BZ + OZ = 160]$$

```
> constraints:=[op(supply),op(demand)];
```

$$\mathit{constraints} := [AW + AX + AY + AZ = 300, BW + BX + BY + BZ = 350,$$
$$OW + OX + OY + OZ = 120, AW + BW + OW = 155,$$
$$AX + BX + OX = 230, AY + BY + OY = 225, AZ + BZ + OZ = 160]$$

```
> LPSolve(cost(AW,AX,AY,AZ,BW,BX,BY,BZ,OW,OX,OY,OZ),constraints,
    assume=nonnegative);
```

$$\big[1.35825\ 10^5, [AW = 0., AX = 110.000000000000, AY = 30.0000000000000,$$
$$AZ = 160.000000000000, BW = 155.000000000000, BX = 0.,$$
$$BY = 195., BZ = 0., OW = 0., OX = 120., OY = 0., OZ = 0.]\big]$$

So an optimal production and shipping plan is as shown in the table below:

	W	X	Y	Z
A		110	30	160
B	155		195	
O		120		

The objective value of this solution (i.e., the total cost of production and shipping) is \$135,825.

(b) Shown below is the initial transportation tableau, with the initial basic feasible solution constructed using the method described in the lecture: satisfy demands one by one, left to right, using the supplies top to bottom. The values of the basic squares are shown in bold in the lower left corners. Then the values of the dual variables are calculated, starting with $v_1 = 0$ arbitrarily, and these dual values are used to compute the test values for the nonbasic squares (shown in red in the lower right corners).

	W: 155	X: 230	Y: 225	Z: 160	
A: 300	$\boxed{155}$ **155**	$\boxed{165}$ **145**	$\boxed{160}$ 0	$\boxed{170}$ 0	$v_1 = 0$
B: 350	$\boxed{170}$ 0	$\boxed{180}$ **85**	$\boxed{175}$ **225**	$\boxed{185}$ **40**	$v_2 = 15$
O: 120	$\boxed{200}$ 0	$\boxed{210}$ 0	$\boxed{205}$ 0	$\boxed{215}$ **120**	$v_3 = 45$
	$w_1 = 155$	$w_2 = 165$	$w_3 = 160$	$w_4 = 170$	

Since all the test values are nonnegative, this solution is already optimal. So we have another optimal production and shipping plan:

	W	X	Y	Z
A	155	145		
B		85	225	40
O				120

The objective value of this solution is 155(\$155) + 145(\$165) + 85(\$180) + 225(\$175) + 40(\$185) + 120(\$215) = \$135,825, which matches the objective value of the optimal solution found by Maple. (In fact, as it turns out, *all* feasible solutions are optimal for this particular instance.) □

4. A manufacturer can ordinarily produce 300 units of a certain product each month and needs to schedule production for a three-month period in which the orders exceed this capacity. Inventory at the beginning of the first month is 120 units, and the demands for the successive three months are 420, 360, and 450 units. Monthly production capacity can be increased by up to 100 units at an additional cost of \$8 per unit. Holding costs to manufacture in one month and ship during a later month are \$2 per unit per month.

Determine a production schedule that will minimize the total cost of exceeding the usual monthly capacity and holding costs.

▷ **Solution.** We can formulate this as an instance of the transportation problem. There are seven origins: with labels and their supplies, they are the starting inventory (I, 120), first month usual production (A, 300), first month increased production (AA, 100), second month usual production (B, 300), second month increased production (BB, 100), third month usual production (C, 300), and third month increased production (CC, 100). Three destinations are explicitly stated in the problem: with labels and their demands, they are the first month (X, 420), second month (Y, 360), and third month (Z, 450). However, the total supply is $120 + 300 + 100 + 300 + 100 + 300 + 100 = 1320$, while the total demand is only $420 + 360 + 450 = 1230$, so we will add a fictitious fourth destination, "unused" (U, 90), to make total supply and total demand equal.

The per-unit costs from each origin to each destination are shown in the following table.

	X	Y	Z	U
I	\$0	\$2	\$4	\$6
A	\$0	\$2	\$4	\$0
AA	\$8	\$10	\$12	\$0
B	∞	\$0	\$2	\$0
BB	∞	\$8	\$10	\$0
C	∞	∞	\$0	\$0
CC	∞	∞	\$8	\$0

Some of the costs in this table are given as ∞, because they represent impossible origin-destination combinations. For example, it is impossible to use ordinary production in the third month (C) to satisfy demand in the first month (X). Because our objective is to minimize total cost, setting the costs of these impossible combinations to ∞ will make them prohibitively expensive to use in an optimal solution. (If ∞ causes implementation difficulties because arithmetic with ∞ is not well defined, then a very large finite cost, such as \$10,000, could be used instead; it would achieve the same result.)

Most of the entries in the U column of this cost table are zero, because they represent origin-destination combinations for which items would not be produced in the first place. For example, if a solution calls for 10 units to be "shipped" from origin AA to destination U, then the meaning is that 10 units that *could* be produced using increased production in the first month will go unused—they will not be used to satisfy demand. In that case, there is no need to produce them at all, so only 90 units will be produced from AA, and those 10 units that are never produced will incur no cost. However, the IU entry in the cost table is \$6, because the items from origin I (the starting inventory) have already been produced, and not using them to satisfy one of the demands will incur three months of holding costs.

The initial transportation tableau appears below.

	X: 420	Y: 360	Z: 450	U: 90	
I: 120	$\boxed{0}$ **120**	$\boxed{2}$ 0	$\boxed{4}$ 0	$\boxed{6}$ 10	$v_1 = 0$
A: 300	$\boxed{0}$ **300**	$\boxed{2}$ 0	$\boxed{4}$ 0	$\boxed{0}$ 4	$v_2 = 0$
AA: 100	$\boxed{8}$ **0**	$\boxed{10}$ **100**	$\boxed{12}$ 0	$\boxed{0}$ -4	$v_3 = 8$
B: 300	$\boxed{\infty}$ ∞	$\boxed{0}$ **260**	$\boxed{2}$ **40**	$\boxed{0}$ 6	$v_4 = -2$
BB: 100	$\boxed{\infty}$ ∞	$\boxed{8}$ 0	$\boxed{10}$ **100**	$\boxed{0}$ -2	$v_5 = 6$
C: 300	$\boxed{\infty}$ ∞	$\boxed{\infty}$ ∞	$\boxed{0}$ **300**	$\boxed{0}$ 8	$v_6 = -4$
CC: 100	$\boxed{\infty}$ ∞	$\boxed{\infty}$ ∞	$\boxed{8}$ **10**	$\boxed{0}$ **90**	$v_7 = 4$
	$w_1 = 0$	$w_2 = 2$	$w_3 = 4$	$w_4 = -4$	

Note that the AAX square has been made basic, with a value of 0, to complete the "stair-step" pattern from the upper left to the lower right. This is important because it gives us the correct number of basic squares and causes the basic squares to correspond to the edges of a tree in the bipartite graph whose vertices are the origins and destinations.

This tableau contains negative test values, so it is not optimal. The most negative test value occurs in the AAU square, so this will be our pivot square. The corresponding pivot circuit goes AAU—CCU—CCZ—BZ—BY—AAY—AAU, so we will add t to the flows in the AAU, CCZ, and BY squares and subtract t from the flows in the CCU, BZ, and AAY squares. Therefore the largest allowable value of t (without causing a negative flow in the CCU, BZ, or AAY square) is $t = 40$. We adjust the flows around the pivot cycle; the flow in the BZ square becomes zero, so that square falls out of the basis. Then we construct the next tableau.

	X: 420	Y: 360	Z: 450	U: 90	
I: 120	$\boxed{0}$ **120**	$\boxed{2}$ 0	$\boxed{4}$ 4	$\boxed{6}$ 14	$v_1 = 0$
A: 300	$\boxed{0}$ **300**	$\boxed{2}$ 0	$\boxed{4}$ 4	$\boxed{0}$ 8	$v_2 = 0$
AA: 100	$\boxed{8}$ **0**	$\boxed{10}$ **60**	$\boxed{12}$ 4	$\boxed{0}$ **40**	$v_3 = 8$
B: 300	$\boxed{\infty}$ ∞	$\boxed{0}$ **300**	$\boxed{2}$ 4	$\boxed{0}$ 10	$v_4 = -2$
BB: 100	$\boxed{\infty}$ ∞	$\boxed{8}$ -4	$\boxed{10}$ **100**	$\boxed{0}$ -2	$v_5 = 10$
C: 300	$\boxed{\infty}$ ∞	$\boxed{\infty}$ ∞	$\boxed{0}$ **300**	$\boxed{0}$ 8	$v_6 = 0$
CC: 100	$\boxed{\infty}$ ∞	$\boxed{\infty}$ ∞	$\boxed{8}$ **50**	$\boxed{0}$ **50**	$v_7 = 8$
	$w_1 = 0$	$w_2 = 2$	$w_3 = 0$	$w_4 = -8$	

The most negative test value in this tableau occurs in the BBY square, for which the pivot circuit is BBY—AAY—AAU—CCU—CCZ—BBZ—BBY. In the pivot we will add t to the flows in the BBY, AAU, and CCZ squares and subtract t from the flows in the AAY, CCU, and BBZ squares, so the largest permissible value of t is $t = 50$. When we adjust the flows, the flow in the CCU square becomes zero, so it falls out of the basis. The next tableau appears below.

	X: 420	Y: 360	Z: 450	U: 90	
I: 120	$\boxed{0}$ **120**	$\boxed{2}$ 0	$\boxed{4}$ 0	$\boxed{6}$ 14	$v_1 = 0$
A: 300	$\boxed{0}$ **300**	$\boxed{2}$ 0	$\boxed{4}$ 0	$\boxed{0}$ 8	$v_2 = 0$
AA: 100	$\boxed{8}$ **0**	$\boxed{10}$ **10**	$\boxed{12}$ 0	$\boxed{0}$ **90**	$v_3 = 8$
B: 300	$\boxed{\infty}$ ∞	$\boxed{0}$ **300**	$\boxed{2}$ 0	$\boxed{0}$ 10	$v_4 = -2$
BB: 100	$\boxed{\infty}$ ∞	$\boxed{8}$ **50**	$\boxed{10}$ **50**	$\boxed{0}$ 2	$v_5 = 6$
C: 300	$\boxed{\infty}$ ∞	$\boxed{\infty}$ ∞	$\boxed{0}$ **300**	$\boxed{0}$ 12	$v_6 = -4$
CC: 100	$\boxed{\infty}$ ∞	$\boxed{\infty}$ ∞	$\boxed{8}$ **100**	$\boxed{0}$ 4	$v_7 = 4$
	$w_1 = 0$	$w_2 = 2$	$w_3 = 4$	$w_4 = -8$	

In this tableau, all test values are nonnegative, so we have reached an optimal solution. In the first month, the manufacturer should use all of the starting inventory and usual production to meet demand and should use increased production to produce an extra 10 units. Those 10 units will be used to satisfy the demand in the second month, along with all of the usual production and 50 extra units produced using increased production in the second month. An additional 50 extra units should also be produced using increased production in the second month to satisfy the demand in the third month, together with all of the usual production and 100 extra units of increased production. The cost of this solution is 120(\$0)+300(\$0)+0(\$8)+10(\$10)+90(\$0)+300(\$0)+50(\$8)+50(\$10)+300(\$0)+100(\$8) = \$1800.

Alternatively, we may formulate this problem as a linear program. For $i \in \{\mathrm{I}, \mathrm{A}, \mathrm{AA}, \mathrm{B}, \mathrm{BB}, \mathrm{C}, \mathrm{CC}\}$ and $j \in \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{U}\}$, let x_{ij} denote the number of units to be taken from origin i and used to satisfy demand at destination j. These variables are all nonnegative. Actually, we do not need the variables x_{BX}, x_{BBX}, x_{CX}, x_{CCX}, x_{CY}, and x_{CCY}, because these origin-destination combinations are impossible. We also do not need the variables x_{AU}, x_{AAU}, x_{BU}, x_{BBU}, x_{CU}, and x_{CCU} if we do not require that total supply equal total demand, and we can remove that requirement by formulating the constraints as inequalities. The linear program is shown below.

$$
\begin{aligned}
\text{minimize} \quad & 2x_{\mathrm{IY}} + 4x_{\mathrm{IZ}} + 6x_{\mathrm{IU}} + 2x_{\mathrm{AY}} + 4x_{\mathrm{AZ}} + 8x_{\mathrm{AAX}} + 10x_{\mathrm{AAY}} \\
& \qquad + 12x_{\mathrm{AAZ}} + 2x_{\mathrm{BZ}} + 8x_{\mathrm{BBY}} + 10x_{\mathrm{BBZ}} + 8x_{\mathrm{CCZ}} \\
\text{subject to} \quad & x_{\mathrm{IX}} + x_{\mathrm{IY}} + x_{\mathrm{IZ}} + x_{\mathrm{IU}} = 120 \\
& x_{\mathrm{AX}} + x_{\mathrm{AY}} + x_{\mathrm{AZ}} \le 300 \\
& x_{\mathrm{AAX}} + x_{\mathrm{AAY}} + x_{\mathrm{AAZ}} \le 100 \\
& x_{\mathrm{BY}} + x_{\mathrm{BZ}} \le 300 \\
& x_{\mathrm{BBY}} + x_{\mathrm{BBZ}} \le 100 \\
& x_{\mathrm{CZ}} \le 300 \\
& x_{\mathrm{CCZ}} \le 100 \\
& x_{\mathrm{IX}} + x_{\mathrm{AX}} + x_{\mathrm{AAX}} \ge 420 \\
& x_{\mathrm{IY}} + x_{\mathrm{AY}} + x_{\mathrm{AAY}} + x_{\mathrm{BY}} + x_{\mathrm{BBY}} \ge 360 \\
& x_{\mathrm{IZ}} + x_{\mathrm{AZ}} + x_{\mathrm{AAZ}} + x_{\mathrm{BZ}} + x_{\mathrm{BBZ}} + x_{\mathrm{CZ}} + x_{\mathrm{CCZ}} \ge 450 \\
& \text{all variables nonnegative.}
\end{aligned}
$$

The following Maple worksheet solves this linear program.

```
> restart;
> with(Optimization);
```

$[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]$

```
> cost:=(IY,IZ,IU,AY,AZ,AAX,AAY,AAZ,BZ,BBY,BBZ,CCZ)->2*IY+4*IZ+6*IU
  +2*AY+4*AZ+8*AAX+10*AAY+12*AAZ+2*BZ+8*BBY+10*BBZ+8*CCZ;
```

$$cost := (IY, IZ, IU, AY, AZ, AAX, AAY, AAZ, BZ, BBY, BBZ, CCZ) \to 2\,IY + 4\,IZ + 6\,IU + 2\,AY + 4\,AZ + 8\,AAX + 10\,AAY + 12\,AAZ + 2\,BZ + 8\,BBY + 10\,BBZ + 8\,CCZ$$

```
> supply:=[IX+IY+IZ+IU=120,AX+AY+AZ<=300,AAX+AAY+AAZ<=100,
  BY+BZ<=300,BBY+BBZ<=100,CZ<=300,CCZ<=100];
```

$$supply := [IX + IY + IZ + IU = 120, AX + AY + AZ \leq 300, AAX + AAY + AAZ \leq 100, BY + BZ \leq 300, BBY + BBZ \leq 100, CZ \leq 300, CCZ \leq 100]$$

```
> demand:=[IX+AX+AAX>=420,IY+AY+AAY+BY+BBY>=360,
  IZ+AZ+AAZ+BZ+BBZ+CZ+CCZ>=450];
```

$$demand := [420 \leq IX + AX + AAX, 360 \leq IY + AY + AAY + BY + BBY, 450 \leq IZ + AZ + AAZ + BZ + BBZ + CZ + CCZ]$$

```
> constraints:=[op(supply),op(demand)];
```

$$constraints := [IX + IY + IZ + IU = 120, AX + AY + AZ \leq 300, AAX + AAY + AAZ \leq 100, BY + BZ \leq 300, BBY + BBZ \leq 100, CZ \leq 300, CCZ \leq 100, 420 \leq IX + AX + AAX, 360 \leq IY + AY + AAY + BY + BBY, 450 \leq IZ + AZ + AAZ + BZ + BBZ + CZ + CCZ]$$

```
> LPSolve(cost(IY,IZ,IU,AY,AZ,AAX,AAY,AAZ,BZ,BBY,BBZ,CCZ),
  constraints,assume=nonnegative);
```

$$\big[1799.99999999174, [AAX = 0., AAY = 0., AAZ = 9.99999999896738, AX = 300., AY = 1.03259778874000\ 10^{-9}, AZ = 0., BBY = 59.9999999989674, BBZ = 40.0000000010326, BY = 300., BZ = 0., CCZ = 100., CZ = 300., IU = 0., IX = 120., IY = 0., IZ = 0.]\big]$$

The optimal solution found by Maple is shown in the table below.

	X	Y	Z
I	120		
A	300		
AA			10
B		300	
BB		60	40
C			300
CC			100

This solution also has cost $1800. □