Combinatorial Optimization

Problem set 3: solutions

1. In class (and on the “Analysis of a simplex tableau” handout), I claimed that if a simplex tableau (for a non-degenerate linear program) contains a column having a negative entry in the objective row and no positive entries below, then the linear program is unbounded. Prove this claim.

▷ **Solution.** Consider a simplex tableau with the variables $x_1, \ldots, x_n$ and objective function z. Let

- $\tilde{A} = [\tilde{a}_{ij}]$ be the matrix of entries in the body of the tableau, below the objective row and to the left of the RHS column, not including the z column;
- $\tilde{b} = [\tilde{b}_1, \ldots, \tilde{b}_m]^\mathsf{T}$ be the column vector of entries in the RHS column below the objective row;
- $\tilde{c}^\mathsf{T} = [\tilde{c}_1, \ldots, \tilde{c}_n]$ be the row vector of entries in the objective row to the left of the RHS column, not including the z column;
- x be the column vector $x = [x_1, \ldots, x_n]^\mathsf{T}$; and
- ζ be the entry in the objective row and the RHS column.

Without loss of generality, we may assume that the basis consists of the variables $x_1, \ldots, x_m$, that the columns $\tilde{A}_1, \ldots, \tilde{A}_m$ form the $m \times m$ identity matrix, and that the x_{m+1} column is the one referred to in the problem, having a negative entry in the objective row with no positive entries below it; if not, we can rearrange the columns and relabel the variables to make this true.

So the tableau looks like this:

x_1	x_2	x_3	$\ldots$	x_m	x_{m+1}	x_{m+2}	x_{m+3}	$\ldots$	x_n	z	RHS
0	0	0	$\ldots$	0	$\tilde{c}_{m+1} < 0$	$\tilde{c}_{m+2}$	$\tilde{c}_{m+3}$	$\ldots$	$\tilde{c}_n$	1	ζ
1	0	0	$\ldots$	0	$\tilde{a}_{1(m+1)} \leq 0$	$\tilde{a}_{1(m+2)}$	$\tilde{a}_{1(m+3)}$	$\ldots$	$\tilde{a}_{1n}$	0	$\tilde{b}_1$
0	1	0	$\ldots$	0	$\tilde{a}_{2(m+1)} \leq 0$	$\tilde{a}_{2(m+2)}$	$\tilde{a}_{2(m+3)}$	$\ldots$	$\tilde{a}_{2n}$	0	$\tilde{b}_2$
0	0	1	$\ldots$	0	$\tilde{a}_{3(m+1)} \leq 0$	$\tilde{a}_{3(m+2)}$	$\tilde{a}_{3(m+3)}$	$\ldots$	$\tilde{a}_{3n}$	0	$\tilde{b}_3$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
0	0	0	$\ldots$	1	$\tilde{a}_{m(m+1)} \leq 0$	$\tilde{a}_{m(m+2)}$	$\tilde{a}_{m(m+3)}$	$\ldots$	$\tilde{a}_{mn}$	0	$\tilde{b}_m$

This tableau represents the system of equations

$$\begin{aligned} \tilde{c}^\mathsf{T} x + z &= \zeta, \\ \tilde{A}x \qquad &= \tilde{b}, \end{aligned}$$

and the system $\tilde{A}x = \tilde{b}$ is equivalent to the original set of constraints in the linear program.

From the equation $\tilde{c}^\mathsf{T} x + z = \zeta$ represented by the objective row, we see that

$$z = \zeta - \sum_{i=1}^{n} \tilde{c}_i x_i. \tag{1}$$

From the system of equations $\tilde{A}x = \tilde{b}$ represented by the body of the tableau, and from the assumption that the columns $\tilde{A}_1, \ldots, \tilde{A}_m$ form the identity matrix, we see that

$$x_i = \tilde{b} - \sum_{j=m+1}^{n} \tilde{a}_{ij} x_j \qquad \text{for } 1 \leq i \leq m. \tag{2}$$

The value of x_{m+1} in the basic feasible solution corresponding to this tableau is zero, because x_{m+1} is nonbasic. Suppose we change the value of x_{m+1} to $t > 0$. Then, by equation (2), in order to satisfy the constraints of the linear program, the value of x_i (for $1 \le i \le m$) must increase by $-\tilde{a}_{i(m+1)}t$, which is nonnegative because $\tilde{a}_{i(m+1)} \le 0$. So the resulting solution will have all nonnegative entries, and therefore it will be feasible. Moreover, by equation (1), the objective value will increase by $-\tilde{c}_{m+1}t$, which is strictly positive because $\tilde{c}_{m+1} < 0$ and can be made arbitrarily large by making t sufficiently large. Hence the linear program has feasible solutions with arbitrarily large objective value, i.e., it is unbounded. □

2. Here is a shortcut for determining the entries in the artificial objective row in an initial (two-phase) simplex tableau:

1. Fill in the entries in the objective row and the rows that come from the constraints.
2. In the columns for artificial variables, enter zeroes in the artificial objective row. (Also, if you are including the $-\xi$ column, enter 1 in that column in the artificial objective row.)
3. In every other column, compute the sum of the entries in the rows that come from constraints having artificial variables (i.e., the rows that come from $\ge$ and $=$ constraints). Negate this sum and enter it in the artificial objective row.

Justify this shortcut.

▷ **Solution.** There's really nothing to justify in step 1, because clearly those rows have to be filled in anyway.

The artificial columns are to be basic in the initial tableau (that is their entire purpose), so the entries in the artificial objective row in columns corresponding to artificial variables must be zero. Likewise, $-\xi$ is to be basic, associated with the artificial objective row, so its column must have 1 in the artificial objective row. This justifies step 2.

Let G denote the set of indices of rows that come from $\ge$ constraints, and let E be those that come from $=$ constraints. So step 3 says that each entry in the initial artificial objective row, except the entries in columns corresponding to artificial variables and to $-\xi$, should be the negation of the sum of the entries in that column in rows $G \cup E$.

The initial artificial objective row represents an equation that is equivalent to

$$\sum_{i \in G \cup E} a_i + (-\xi) = 0, \tag{3}$$

but with $\sum_{i \in G \cup E} a_i$ expressed in terms of nonbasic variables (i.e., variables other than slack and artificial variables).

For $g \in G$, the gth row of the body of the initial tableau represents the equation

$$\sum_{j=1}^{n} a_{gj} x_j - p_g + a_g = b_g.$$

For $e \in E$, the eth row of the body of the initial tableau represents the equation

$$\sum_{j=1}^{n} a_{ej} x_j + a_e = b_e.$$

Solving these equations for a_g and a_e, respectively, yields

$$a_g = b_g - \sum_{j=1}^{n} a_{gj} x_j + p_g \qquad \text{for } g \in G,$$
$$a_e = b_e - \sum_{j=1}^{n} a_{ej} x_j \qquad \text{for } e \in E.$$

So

$$\sum_{i \in G \cup E} a_i = \sum_{i \in G \cup E} b_i - \sum_{j=1}^{n} \sum_{i \in G \cup E} a_{ij} x_j + \sum_{g \in G} p_g,$$

and hence equation (3) becomes

$$-\sum_{j=1}^{n} \sum_{i \in G \cup E} a_{ij} x_j + \sum_{g \in G} p_g + (-\xi) = -\sum_{i \in G \cup E} b_i. \tag{4}$$

In this equation:

- The coefficient of x_j, $-\sum_{i \in G \cup E} a_{ij}$, is the negation of the sum of the entries in the x_j column of the initial tableau in the rows $G \cup E$.
- The coefficient of any slack variable s_j is zero, and the entries in the s_j column in the s_j column of the initial tableau in the rows $G \cup E$ are all zero (because $\geq$ and $=$ constraints do not have slack variables).
- The coefficient of any surplus variable p_j is 1, and the entries in the p_j column in the p_j column of the initial tableau in the rows $G \cup E$ are all zero except for a single -1 entry in the row that comes from the $\geq$ constraint containing p_j.
- The coefficient of every artificial variable is zero.
- The coefficient of $-\xi$ is 1.
- The right-hand side is $-\sum_{i \in G \cup E} b_i$, which is the negation of the sum of the entries in the RHS column of the initial tableau in the rows $G \cup E$.

So equation (4) expresses equation (3) in terms of nonbasic variables (all slack and artificial variables have coefficient zero), and each coefficient other than those for artificial variables and $-\xi$ is the negation of the sum of the entries in that column in the initial tableau in the rows $G \cup E$, which justifies step 3. □

3. Solve the following linear program by hand, using the two-phase simplex algorithm.

$$\begin{aligned} \text{maximize} \quad & 3x_1 - 8x_2 + 10x_3 \\ \text{subject to} \quad & x_1 + x_2 + 3x_3 \leq 40 \\ & 5x_1 - x_3 \geq 10 \\ & 2x_1 - x_2 + x_3 = 12 \\ & x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \text{ unrestricted.} \end{aligned}$$

▷ **Solution.** First we convert to standard variable domains. Let $\bar{x}_2 = -x_2$ and let $x_3 = x_3^+ - x_3^-$, where $x_3^+ \geq 0$ and $x_3^- \geq 0$. After we make these substitutions, the linear program becomes

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 8\bar{x}_2 + 10x_3^+ - 10x_3^- \\ \text{subject to} \quad & x_1 - \bar{x}_2 + 3x_3^+ - 3x_3^- \leq 40 \\ & 5x_1 - x_3^+ + x_3^- \geq 10 \\ & 2x_1 + \bar{x}_2 + x_3^+ - x_3^- = 12 \\ & x_1 \geq 0, \quad \bar{x}_2 \geq 0, \quad x_3^+ \geq 0, \quad x_3^- \geq 0. \end{aligned}$$

Now we insert slack, surplus, and artificial variables:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 8\bar{x}_2 + 10x_3^+ - 10x_3^- \\ \text{subject to} \quad & x_1 - \bar{x}_2 + 3x_3^+ - 3x_3^- + s_1 = 40 \\ & 5x_1 - x_3^+ + x_3^- - p_2 + a_2 = 10 \\ & 2x_1 + \bar{x}_2 + x_3^+ - x_3^- + a_3 = 12 \\ & x_1 \geq 0, \quad \bar{x}_2 \geq 0, \quad x_3^+ \geq 0, \quad x_3^- \geq 0, \quad s_1 \geq 0, \quad p_2 \geq 0, \quad a_2 \geq 0, \quad a_3 \geq 0. \end{aligned}$$

We begin Phase I, using the artificial objective row to guide our pivots. The sequence of simplex tableaux for Phase I appears below. The pivot entry is indicated in each tableau.

x_1	$\bar{x}_2$	x_3^+	x_3^-	s_1	p_2	a_2	a_3	z	$-\xi$	RHS
-3	-8	-10	10	0	0	0	0	1	0	0
-7	-1	0	0	0	1	0	0	0	1	-22
1	-1	3	-3	1	0	0	0	0	0	40
$\boxed{5}$	0	-1	1	0	-1	1	0	0	0	10
2	1	1	-1	0	0	0	1	0	0	12

x_1	$\bar{x}_2$	x_3^+	x_3^-	s_1	p_2	a_2	a_3	z	$-\xi$	RHS
0	-8	$-53/5$	$53/5$	0	$-3/5$	$3/5$	0	1	0	6
0	-1	$-7/5$	$7/5$	0	$-2/5$	$7/5$	0	0	1	-8
0	-1	$16/5$	$-16/5$	1	$1/5$	$-1/5$	0	0	0	38
1	0	$-1/5$	$1/5$	0	$-1/5$	$1/5$	0	0	0	2
0	1	$\boxed{7/5}$	$-7/5$	0	$2/5$	$-2/5$	1	0	0	8

x_1	$\bar{x}_2$	x_3^+	x_3^-	s_1	p_2	a_2	a_3	z	$-\xi$	RHS
0	$-3/7$	0	0	0	$17/7$	$-17/7$	$53/7$	1	0	$466/7$
0	0	0	0	0	0	1	1	0	1	0
0	$-23/7$	0	0	1	$-5/7$	$5/7$	$-16/7$	0	0	$138/7$
1	$1/7$	0	0	0	$-1/7$	$1/7$	$1/7$	0	0	$22/7$
0	$5/7$	1	-1	0	$2/7$	$-2/7$	$5/7$	0	0	$40/7$

After two pivots, we reach a tableau that has no negative entries in the artificial objective row to the left of the RHS column, so Phase I is complete. The value of ξ in this tableau is 0, so Phase I was successful. All artificial variables are nonbasic, so we can delete the artificial objective row and the columns for the artificial variables and $-\xi$ and go on to Phase II. The sequence of simplex tableaux for Phase II appears below.

x_1	$\bar{x}_2$	x_3^+	x_3^-	s_1	p_2	z	RHS
0	$-3/7$	0	0	0	$17/7$	1	$466/7$
0	$-23/7$	0	0	1	$-5/7$	0	$138/7$
1	$1/7$	0	0	0	$-1/7$	0	$22/7$
0	$\boxed{5/7}$	1	-1	0	$2/7$	0	$40/7$

x_1	$\bar{x}_2$	x_3^+	x_3^-	s_1	p_2	z	RHS
0	0	$3/5$	$-3/5$	0	$13/5$	1	70
0	0	$23/5$	$-23/5$	1	$3/5$	0	46
1	0	$-1/5$	$\boxed{1/5}$	0	$-1/5$	0	2
0	1	$7/5$	$-7/5$	0	$2/5$	0	8

x_1	$\bar{x}_2$	x_3^+	x_3^-	s_1	p_2	z	RHS
3	0	0	0	0	2	1	76
23	0	0	0	1	-4	0	92
5	0	-1	1	0	-1	0	10
7	1	0	0	0	-1	0	22

After two more pivots, we reach a tableau that has no negative entries in the objective row to the left of the RHS column, so the corresponding basic basic feasible solution is optimal. This optimal solution is $x_1 = 0$, $\bar{x}_2 = 22$, $x_3^+ = 0$, $x_3^- = 10$, $s_1 = 92$, $p_2 = 0$. In terms of the original variables, we have

$$x_1 = 0, \quad x_2 = -22, \quad x_3 = -10$$

(and $s_1 = 92$, $p_2 = 0$). The optimal objective value is 76. □

4. In the description of the two-phase simplex algorithm in class, I omitted one possibility that may occur at the end of Phase I: the value of ξ is 0, but at least one artificial variable remains in the basis (having the value 0). If this happens, then we need to "drive the artificial variable out of the basis" in order to get a basis consisting solely of non-artificial variables, so that we can begin Phase II. Papadimitriou and Steiglitz discuss this case (which they call "Case 3") in Section 2.8, on page 56. Give an example of a linear program for which this case occurs, and go through the full two-phase simplex algorithm to solve your example.

▷ **Solution.** A very simple (and therefore pretty stupid) example is the linear program

$$\begin{aligned} \text{maximize} \quad & x \\ \text{subject to} \quad & -x = 0 \\ & x \geq 0. \end{aligned}$$

When we insert an artificial variable into the = constraint, we get

$$\begin{aligned} \text{maximize} \quad & x \\ \text{subject to} \quad & -x + a_1 = 0 \\ & x \geq 0, \quad a_1 \geq 0. \end{aligned}$$

The initial (two-phase) simplex tableau is

x	a_1	z	$-\xi$	RHS
-1	0	1	0	0
1	0	0	1	0
-1	1	0	0	0

Phase I is done already. The value of ξ is 0, so Phase I was successful. However, the artificial variable a_1 is still basic. So we need to drive the artificial variable out of the basis. We do this by pivoting on any nonzero (not necessarily positive) entry in the row corresponding to a_1 and in a column corresponding to a non-artificial variable:

x	a_1	z	$-\xi$	RHS
-1	0	1	0	0
1	0	0	1	0
$\boxed{-1}$	1	0	0	0

x	a_1	z	$-\xi$	RHS
0	-1	1	0	0
0	1	0	1	0
1	-1	0	0	0

Now a_1 is nonbasic, so we can delete the artificial objective row and the columns for a_1 and $-\xi$ and continue with Phase II:

x	z	RHS
0	1	0
1	0	0

Phase II is done immediately, and the optimal solution is (unsurprisingly) $x = 0$, with an optimal objective value of 0. □

5. Write the dual of the following linear program.

$$\begin{aligned} \text{maximize} \quad & x_1 - 2x_2 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + x_4 \geq 0 \\ & 4x_1 + 3x_2 - 4x_3 - 2x_4 \leq 3 \\ & -x_1 - x_2 - 2x_3 + x_4 = 1 \\ x_1 \text{ unrestricted}, \quad & x_2 \geq 0, \quad x_3 \leq 0, \quad x_4 \text{ unrestricted.} \end{aligned}$$

The optimal solution of the linear program above has $x_1 = 5/2$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 7/2$. Use complementary slackness to determine the optimal solution to the dual. Verify that the two solutions are both feasible for their respective linear programs and that they have the same objective value.

▷ **Solution.** The dual linear program is

$$\begin{aligned} \text{minimize} \quad & 3y_2 + y_3 \\ \text{subject to} \quad & y_1 + 4y_2 - y_3 = 1 \\ & 2y_1 + 3y_2 - y_3 \geq -2 \\ & y_1 - 4y_2 - 2y_3 \leq 0 \\ & y_1 - 2y_2 + y_3 = 0 \\ y_1 \leq 0, \quad & y_2 \geq 0, \quad y_3 \text{ unrestricted.} \end{aligned}$$

First we apply complementary slackness to the primal constraints and the dual variables.

1. Either $x_1 + 2x_2 + x_3 + x_4 = 0$ or $y_1 = 0$ (or both). The first equation is false, because $5/2 + 2(0) + 0 + 7/2 = 6 \neq 0$, so we can conclude that $y_1 = 0$.
2. Either $4x_1 + 3x_2 - 4x_3 - 2x_4 = 3$ or $y_2 = 0$ (or both). The first equation is true, because $4(5/2) + 3(0) - 4(0) - 2(7/2) = 3$, so we cannot conclude anything about y_2.
3. Either $-x_1 - x_2 - 2x_3 + x_4 = 1$ or $y_3 = 0$ (or both). The first equation is true, because $-(5/2) - 0 - 2(0) + 7/2 = 1$, so we cannot conclude anything about y_3.

Then we apply complementary slackness to the dual constraints and the primal variables.

1. Either $y_1 + 4y_2 - y_3 = 1$ or $x_1 = 0$ (or both). The second equation is false, because $x_1 = 5/2 \neq 0$, so we can conclude that $y_1 + 4y_2 - y_3 = 1$.
2. Either $2y_1 + 3y_2 - y_3 = -2$ or $x_2 = 0$ (or both). The second equation is true, because $x_2 = 0$, so we cannot conclude anything else.
3. Either $y_1 - 4y_2 - 2y_3 = 0$ or $x_3 = 0$ (or both). The second equation is true, because $x_3 = 0$, so we cannot conclude anything else.
4. Either $y_1 - 2y_2 + y_3 = 0$ or $x_4 = 0$ (or both). The second equation is false, because $x_4 = 7/2 \neq 0$, so we can conclude that $y_1 - 2y_2 + y_3 = 0$.

Therefore we know

$$\begin{cases} y_1 = 0, \\ y_1 + 4y_2 - y_3 = 1, \\ y_1 - 2y_2 + y_3 = 0. \end{cases}$$

The solution to this system gives us the values of the dual variables: $y_1 = 0$, $y_2 = 1/2$, and $y_3 = 1$.

We verify that the given solution ($x_1 = 5/2$, $x_2 = 0$, $x_3 = 0$, $x_4 = 7/2$) is feasible in the primal linear program:

- It satisfies the constraints, because

$$\begin{aligned} 5/2 + 2(0) + 0 + 7/2 &= 6 \geq 0, \\ 4(5/2) + 3(0) - 4(0) - 2(7/2) &= 3 \leq 3, \quad \text{and} \\ -5/2 - 0 - 2(0) + 7/2 &= 1 = 1. \end{aligned}$$

- It satisfies the variable domains, because $x_2 = 0 \geq 0$ and $x_3 = 0 \leq 0$. (The values of x_1 and x_4 are unrestricted.)

We also verify that the solution we found to the dual linear program ($y_1 = 0$, $y_2 = 1/2$, $y_3 = 1$) is feasible:

- It satisfies the constraints, because

$$\begin{aligned} 0 + 4(1/2) - 1 = 1 &= 1, \\ 2(0) + 3(1/2) - 1 = 1/2 &\geq -2, \\ 0 - 4(1/2) - 2(1) = -4 &\leq 0, \quad \text{and} \\ 0 - 2(1/2) + 1 = 0 &= 0. \end{aligned}$$

- It satisfies the variable domains, because $y_1 = 0 \leq 0$ and $y_2 = 1/2 \geq 0$. (The value of y_3 is unrestricted.)

The objective value of the primal solution is $x_1 - 2x_2 = 5/2 - 2(0) = 5/2$, and the objective value of the dual solution is $3y_2 + y_3 = 3(1/2) + 1 = 5/2$, so the two solutions have the same objective value. □

This wasn't part of the assigned problem, but perhaps it should have been:

Note that the optimal values of the dual variables are the coefficients in a linear combination of the constraints in the primal linear program that proves the optimality of the primal solution:

$$\begin{array}{l} 0(\;\;x_1 + 2x_2 + \;\;x_3 + \;\;x_4 \;\;\geq 0) \\ \frac{1}{2}(\;4x_1 + 3x_2 - 4x_3 - 2x_4 \;\;\leq 3) \\ +\,1(-x_1 - \;\;x_2 - 2x_3 + \;\;x_4 \;\;= 1) \\ \hline 0(\;\;x_1 + 2x_2 + \;\;x_3 + \;\;x_4) \leq 0(0) \\ \frac{1}{2}(\;4x_1 + 3x_2 - 4x_3 - 2x_4) \leq \frac{1}{2}(3) \\ +\,1(-x_1 - \;\;x_2 - 2x_3 + \;\;x_4) = 1(1) \\ \hline \;\;x_1 + \frac{1}{2}x_2 \qquad\qquad\qquad \leq \frac{5}{2} \end{array}$$

Since $x_2 \geq 0$, we conclude from this inequality that every feasible value of the primal objective function z must satisfy

$$z = x_1 - 2x_2 \leq x_1 + \tfrac{1}{2}x_2 \leq \tfrac{5}{2}.$$

Therefore, the feasible solution $x = [5/2, 0, 0, 7/2]^\mathsf{T}$, which yields the objective value $5/2$, must be optimal.

Likewise, the optimal values of the primal variables are the coefficients in a linear combination of the constraints in the dual linear program that proves the optimality of the dual solution:

$$
\begin{array}{rrcl}
 & \frac{5}{2}(\ y_1 + 4y_2 - \ \ y_3 & = & \ \ 1) \\
 & 0(2y_1 + 3y_2 - \ \ y_3 & \geq & -2) \\
 & 0(\ y_1 - 4y_2 - 2y_3 & \leq & \ \ 0) \\
+ & \frac{7}{2}(\ y_1 - 2y_2 + \ \ y_3 & = & \ \ 0 \\
\hline
 & \frac{5}{2}(\ y_1 + 4y_2 - \ \ y_3) & = & \frac{5}{2}(1) \\
 & 0(2y_1 + 3y_2 - \ \ y_3) & \geq & 0(-2) \\
 & 0(\ y_1 - 4y_2 - 2y_3) & \geq & 0(0) \\
+ & \frac{7}{2}(\ y_1 - 2y_2 + \ \ y_3) & = & \frac{7}{2}(0) \\
\hline
 & 3y_1 + 3y_2 + \ \ y_3 & \geq & \frac{5}{2}
\end{array}
$$

Since $y_1 \leq 0$, we conclude from this inequality that every feasible value of the dual objective function w must satisfy

$$w = 3y_2 + y_3 \geq 3y_1 + 3y_2 + y_3 \geq \tfrac{5}{2}.$$

Therefore, the feasible solution $y = [0, 1/2, 1]^{\mathsf{T}}$, which yields the objective value 5/2, must be optimal.

6. Describe (at least) two essentially different ways to use the (maximizing) simplex algorithm to solve a minimization linear program. What are the comparative advantages and disadvantages of each? Given a minimization linear program, what characteristics would indicate that one method or the other may be a better approach?

▷ **Solution.** One method is simply to negate the objective function. Maximizing the negation is equivalent to minimizing the original function. The main advantage of this method is that it is simple and direct to understand and implement. One disadvantage is that minimization LPs canonically have $\geq$ constraints, which will require the use of the two-phase simplex algorithm.

Another method is to solve the *dual* maximization LP and then use complementary slackness to solve for the optimal primal solution. One advantage of this approach is that the dual of a minimization LP in canonical form is a maximization LP in canonical form, which does not require the two-phase simplex algorithm (as long as the right-hand sides of the constraints are nonnegative, i.e., if the primal minimization LP has nonnegative coefficients in the objective function). A disadvantage of this method is that it requires the computation of the dual LP; in addition to requiring additional time to write the dual, this may introduce the complications of nonstandard variable domains if the constraints in the primal LP are not in canonical form. Another disadvantage is the need to go through the reasoning of complementary slackness and to solve a system of equations to obtain the primal solution.

A third method is to solve the dual maximization LP using the simplex algorithm, keeping the artificial variable columns (if any) through Phase II, and then read off the solution to the primal minimization LP from the optimal tableau as the entries in the objective row in the columns corresponding to slack and artificial variables. This method shares with the second method the advantage that the dual of a minimization LP in canonical form is a maximization LP in canonical form and the disadvantage that the dual must be computed. However, an additional advantage is that we get the optimal primal solution immediately upon solving the dual; no further work is necessary. On the other hand, an additional disadvantage is the need to carry the artificial variable columns through Phase II (and to remember not to pivot on them!).

If the minimization LP has many $\leq$ constraints (with nonnegative right-hand sides), then simply negating the objective function might be a good approach, because such constraints in a maximization LP are easily handled directly by the simplex algorithm. If the minimization LP is in canonical form and has nonnegative objective function coefficients, then one of the two approaches using the dual LP is probably best, because it will not be necessary to use the two-phase simplex algorithm to solve the dual. The third approach is likely to be computationally better than the second, because carrying the artificial variable columns through the pivots really is equivalent to solving the system obtained from complementary slackness using row operations, so these steps might as well be done together. On the other hand, the sequence of pivots taken in the simplex algorithm may not be the most *efficient* sequence of row operations to solve that system, in which case the second approach may be better (but this is not something that can be easily predicted ahead of time). □