Combinatorial Optimization

Problem set 2

Assigned Monday, June 1, 2015. Due Thursday, June 4, 2015.

1. Consider the following two linear programs in standard form:

$$\begin{array}{rl} \text{maximize} & c^{\mathrm{T}}x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad\qquad \begin{array}{rl} \text{maximize} & -c^{\mathrm{T}}x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Can both of these linear programs have feasible solutions with arbitrarily large objective value? If yes, give an example; if not, prove so.

2. In class we saw an example that served as a sketch of a proof of the following theorem:

> **Theorem.** *Let x be a feasible solution to a maximizing linear program (in standard form). Then either there exists a basic feasible solution whose objective value is at least as large as that of x, or else the linear program is unbounded.*

Using the example as a guide, prove this theorem.

3. Convert the following linear program to standard form. Write the initial simplex tableau and give the initial basic feasible solution. Do a pivot to bring x_2 into the basis and give the resulting basic feasible solution.

$$\begin{array}{rl} \text{maximize} & 5x_1 + 3x_2 - 2x_3 \\ \text{subject to} & x_1 + 2x_2 + x_3 \leq 10 \\ & 4x_1 + 5x_2 \leq 20 \\ & 2x_1 - 3x_2 + 2x_3 \leq 6 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{array}$$

4. Answer yes or no and prove your answer: Can a pivot move the corresponding basic feasible solution a positive distance in $\mathbb{R}^n$ while leaving the objective value unchanged?

5. Solve the following linear program by hand, using the simplex algorithm.

$$\begin{array}{rl} \text{maximize} & 20x_1 + 6x_2 + 8x_3 \\ \text{subject to} & 6x_1 + 2x_2 + 3x_3 \leq 420 \\ & 4x_1 + 3x_2 \leq 200 \\ & x_3 \leq 50 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{array}$$

6. Consider the following simplex tableau (for the maximizing simplex algorithm).

x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	RHS
a_1	0	0	0	40	0	25	1	700
a_2	0	0	1	1/2	0	-2	0	84
a_3	1	0	0	-2	0	5/2	0	225
a_4	0	1	0	3/2	0	1/2	0	125
a_5	0	0	0	$-5/2$	1	$-3/2$	0	65

For each part below, describe conditions on the entries $a_1, \ldots, a_5$ in the first column so that the tableau satisfies the stated condition. Try to make your answers as general as possible.

(a) The corresponding basic feasible solution is optimal and unique.

(b) The corresponding basic feasible solution is optimal but not unique, and $x_1 = 13$ in the alternative optimal basic feasible solution.

(c) The corresponding basic feasible solution is not optimal, and in the next basic feasible solution (that is, the basic feasible solution corresponding to the tableau after the next pivot in the simplex algorithm), $s_1 = 0$ and the value of the objective function is 742.

(d) The corresponding basic feasible solution is not optimal, and in the next basic feasible solution $x_3 = 0$ and $s_3 = 40$.