Combinatorial Optimization

Problem set 1: solutions

1. Formulate a linear program for the following optimization problem. Then solve your linear program with Maple and interpret the results.

> The Ace Refining Company produces two types of unleaded gasoline, regular and premium, which it sells to its chain of service stations for \$36 and \$42 per barrel, respectively. Both types are blended from Ace's inventories of refined domestic and refined foreign oil and must meet the following specifications:
>
> | | Maximum vapor pressure | Minimum octane rating | Maximum demand, bbl/wk | Minimum deliveries, bbl/wk |
> |---|---|---|---|---|
> | Regular | 23 | 88 | 100,000 | 50,000 |
> | Premium | 23 | 93 | 20,000 | 5,000 |
>
> The characteristics of the refined oils in inventory are:
>
> | | Vapor pressure | Octane rating | Inventory, bbl | Cost, \$/bbl |
> |---|---|---|---|---|
> | Domestic | 25 | 87 | 40,000 | 16 |
> | Foreign | 15 | 98 | 60,000 | 30 |
>
> What quantities of the two oils should Ace blend into the two gasolines in order to maximize weekly profit? (Assume that vapor pressure and octane ratings combine linearly in a blend.)

Solution. Let DR and FR denote the numbers of barrels of domestic and foreign oils, respectively, to be blended into regular gasoline per week, and let DP and FP denote the numbers of barrels of domestic and foreign oils, respectively, to be blended into premium gasoline per week. (One nice thing about linear programming is that since everything is linear we can have multicharacter variable names without ambiguity, because variables are never multiplied together in a linear program.) All of these variables are nonnegative.

Ace's weekly profit is given by

$$(36-16)DR+(36-30)FR+(42-16)DP+(42-30)FP = 20DR+6FR+26DP+12FP.$$

This is the objective function; the objective is to maximize this value.

Under the assumption that vapor pressure combines linearly in a blend, the vapor pressure of the resulting regular gasoline will be $(25DR+15FR)/(DR+FR)$, so to meet the maximum vapor pressure constraint for regular gasoline we need

$$\frac{25DR+15FR}{DR+FR} \le 23.$$

As written, this constraint is not linear. But we can multiply both sides by $DR+FR$ to get $25DR+15FR \le 23DR+23FR$, or $2DR-8FR \le 0$. Likewise, the maximum vapor pressure constraint for premium gasoline can be expressed as $2DP-8FP \le 0$.

Similarly, the minimum octane rating constraints for regular and premium gasoline, respectively, are

$$\frac{87DR+98FR}{DR+FR} \ge 88 \qquad \text{and} \qquad \frac{87DP+98FP}{DP+FP} \ge 93.$$

When written as linear constraints, these become $-DR+10FR \ge 0$ and $-6DP+5FP \ge 0$.

After including constraints for maximum demand, minimum deliveries, and inventory, we get the following linear program.

$$
\begin{array}{lrcl}
\text{maximize} & 20DR + 6FR + 26DP + 12FP & & & \text{[weekly profit]} \\
\text{subject to} & 2DR - 8FR & \le & 0 & \text{[vapor pressure, regular]} \\
& 2DP - 8FP & \le & 0 & \text{[vapor pressure, premium]} \\
& -DR + 10FR & \ge & 0 & \text{[octane rating, regular]} \\
& -6DP + 5FP & \ge & 0 & \text{[octane rating, premium]} \\
& DR + FR & \le & 100{,}000 & \text{[max. demand, regular]} \\
& DR + FR & \ge & 50{,}000 & \text{[min. deliveries, regular]} \\
& DP + FP & \le & 20{,}000 & \text{[max. demand, premium]} \\
& DP + FP & \ge & 5{,}000 & \text{[min. deliveries, premium]} \\
& DR + DP & \le & 40{,}000 & \text{[inventory, domestic]} \\
& FR + FP & \le & 60{,}000 & \text{[inventory, foreign]} \\
& \multicolumn{3}{l}{DR \ge 0, \quad FR \ge 0, \quad DP \ge 0, \quad FP \ge 0.} &
\end{array}
$$

The following Maple worksheet can be used to solve this linear program.

```
> restart;
> with(Optimization);
```

$[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]$

```
> f:=(DR,FR,DP,FP)->20*DR+6*FR+26*DP+12*FP;
```

$$f := (DR, FR, DP, FP) \to 20DR + 6FR + 26DP + 12FP$$

```
> constraints:=[2*DR-8*FR<=0,2*DP-8*FP<=0,-DR+10*FR>=0,
    -6*DP+5*FP>=0,DR+FR<=100000,DR+FR>=50000,DP+FP<=20000,
    DP+FP>=5000,DR+DP<=40000,FR+FP<=60000];
```

$$constraints := [2DR - 8FR \le 0, 2DP - 8FP \le 0, 0 \le -DR + 10FR, 0 \le -6DP + 5FP, DR + FR \le 100000, 50000 \le DR + FR, DP+FP \le 20000, 5000 \le DP+FP, DR+DP \le 40000, FR+FP \le 60000]$$

```
> LPSolve(f(DR,FR,DP,FP),constraints,'maximize',
    assume=nonnegative);
```

$$\left[1.280000\ 10^6, [DP = 9090.90909090919, DR = 30909.0909090908, FP = 10909.0909090908, FR = 49090.9090909092]\right]$$

Therefore, an optimal solution is to blend approximately 30,909 gallons of domestic oil and 49,091 gallons of foreign oil to produce regular gasoline, and to blend approximately 9,091 gallons of domestic oil and 10,909 gallons of foreign oil to produce premium gasoline; this will produce a profit of \$1,280,000. (The exact values of the variables in this solution are $DR = 340{,}000/11$, $FR = 540{,}000/11$, $DP = 100{,}000/11$, and $FP = 120{,}000/11$.)

This is not the only optimal solution. There is another optimal basic feasible solution, which is $DR = 40{,}000$, $FR = 40{,}000$, $DP = 0$, and $FP = 20{,}000$. This solution also yields a profit of \$1,280,000. (And any convex combination of these two solutions is also an optimal feasible solution.) □

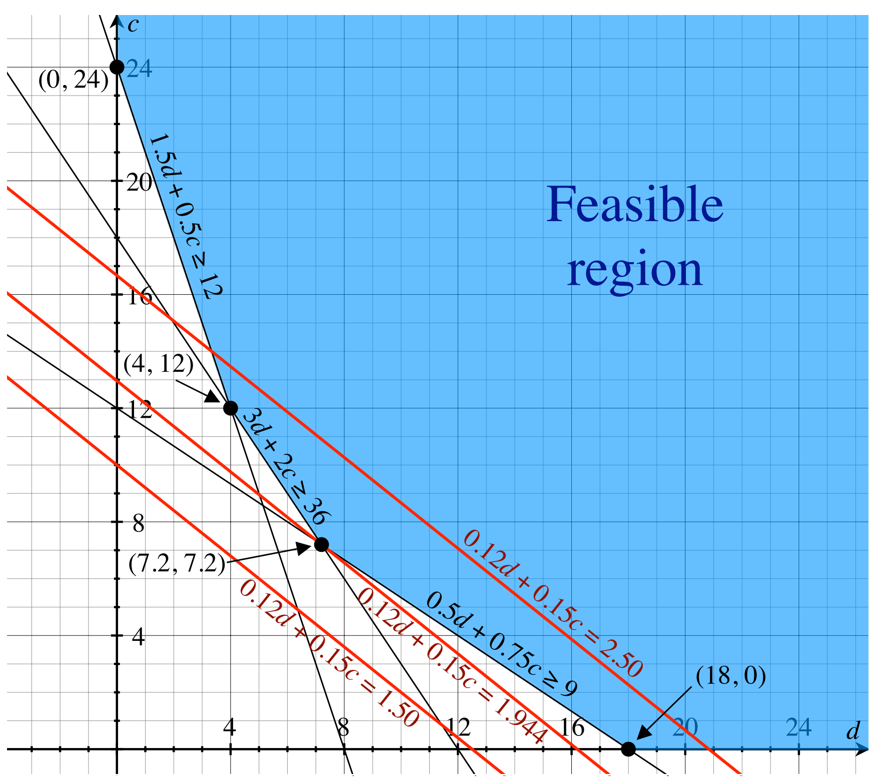
2. Formulate a linear program for the following optimization problem. Carefully draw the feasible region (i.e., the set of feasible solutions). Solve the linear program graphically. Draw at least three level curves of the objective function, including the level curve corresponding to the optimal value.

> A kennel owner has a choice of two dog foods to buy in bulk quantities to feed the dogs under her care. The average dog in her kennel needs at least 36 grams of protein, 12 grams of fat, and 9 milligrams of iron a day. An ounce of Dog Grub includes 3 grams of protein, 1.5 grams of fat, and 0.5 milligrams of iron. An ounce of Canine Chow includes 2 grams of protein, 0.5 grams of fat, and 0.75 milligrams of iron. If Dog Grub and Canine Chow cost \$0.12 and \$0.15 per ounce, respectively, how much of each should she buy to meet the daily needs of the average dog at a minimum cost?

Solution. Let d and c denote the number of ounces of Dog Grub and Canine Chow, respectively, to be bought. Then the following linear program minimizes the total cost subject to the nutritional requirements.

$$\begin{array}{rll} \text{minimize} & 0.12d + 0.15c & \text{[cost]} \\ \text{subject to} & 3\ \ d + 2\ \ c \geq 36 & \text{[protein]} \\ & 1.5\ d + 0.5\ c \geq 12 & \text{[fat]} \\ & 0.5\ d + 0.75c \geq \ 9 & \text{[iron]} \\ & d \geq 0, \quad c \geq 0. & \end{array}$$

The feasible region is shaded in the figure below. It has four corners: $(0, 24)$, $(4, 12)$, $(7.2, 7.2)$, and $(18, 0)$.



To solve this linear program graphically, we evaluate the objective function at each of these corners:

Corner (d, c)	Objective value
$(0, 24)$	3.60
$(4, 12)$	2.28
$(7.2, 7.2)$	1.944
$(18, 0)$	2.16

Since this is a minimizing linear program, we choose the smallest of these objective values. Therefore, the optimal solution is $d = 7.2$, $c = 7.2$, having an objective value of 1.944. In other words, the kennel owner should buy 7.2 ounces of both Dog Grub and Canine Chow for each dog each day, at a total cost of just over \$1.94.

The figure above also shows three level curves of the objective function, corresponding to the objective values 1.50, 1.944, and 2.50. Note that these level curves are parallel lines. The level curve $0.12d + 0.15c = 1.50$ does not pass through the feasible region, indicating that the objective value 1.50 cannot be attained. The level curve $0.12d + 0.15c = 2.50$ does pass through the feasible region, but 2.50 is not the optimal objective value because level curves corresponding to smaller objective values also pass through the feasible region. The level curve $0.12d + 0.15c = 1.944$ corresponds to the optimal objective value; it intersects the feasible region at exactly one point, which is the corner $(7.2, 7.2)$. From the level curves, we see that the objective function decreases as we move down and to the left, so we can conclude that the linear program does have an optimal solution even though the feasible region is unbounded. □

3. Give a (simple) example of each of the following:

(a) An infeasible linear program.
(b) An unbounded linear program (i.e., a linear program with an unbounded objective value).
(c) A linear program with a unique optimal solution.
(d) A linear program with a nonunique optimal solution.
(e) A linear program with an unbounded feasible region that does *not* have an unbounded objective value.

Solution.

(a) Here is an example of an infeasible linear program:

$$\begin{aligned} \text{maximize} \quad & x \\ \text{subject to} \quad & x \leq -1 \\ & x \geq \ \ 0. \end{aligned}$$

(b) Here is an unbounded linear program:

$$\begin{aligned} \text{maximize} \quad & x \\ & x \geq 0. \end{aligned}$$

(Note that the set of constraints here happens to be empty. The inequality $x \geq 0$ is a variable domain, not a constraint.)

(c) The following linear program has a unique optimal solution:

$$\begin{aligned} \text{maximize} \quad & x \\ \text{subject to} \quad & x \leq 1 \\ & x \geq 0. \end{aligned}$$

The unique optimal solution is $x = 1$.

(d) The following linear program has a nonunique optimal solution:

$$\begin{aligned} \text{maximize} \quad & x + y \\ \text{subject to} \quad & x + y \le 1 \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

The optimal objective value, 1, is achieved at all points on the line segment joining $(0,1)$ and $(1,0)$.

(e) Here is a linear program with an unbounded feasible region that does not have an unbounded objective value:

$$\begin{aligned} \text{minimize} \quad & x + y \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

The feasible region is the entire first quadrant, which is an unbounded region, but the optimal objective value is 0, so the *linear program* is not unbounded. (For another example, see Problem 2.) □

4. A *convex combination* of two vectors x and x' (of the same size) is a vector of the form $\lambda x + (1-\lambda)x'$, where λ is a scalar in the interval $[0,1]$. Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $b = [b_1, \dots, b_m]^{\mathrm{T}}$ be an $m \times 1$ column vector. Consider the set of constraints $Ax \le b$, where x is an $n \times 1$ column vector. (The inequality $Ax \le b$ means that each coordinate of Ax is less than or equal to the corresponding coordinate of b; so it is a set of m inequalities.) Suppose that $x = [x_1, \dots, x_n]^{\mathrm{T}}$ and $x' = [x'_1, \dots, x'_n]^{\mathrm{T}}$ are both feasible solutions to this set of constraints. Prove that any convex combination of x and x' is also feasible.

Solution. Let $\lambda \in [0,1]$. Since x and x' are both feasible solutions to this set of constraints, we know $Ax \le b$ and $Ax' \le b$, so $Ax - b \le 0$ and $Ax' - b \le 0$. Therefore

$$\begin{aligned} A[\lambda x + (1-\lambda)x'] - b &= \lambda Ax + (1-\lambda)Ax' - \lambda b - (1-\lambda)b \\ &= \underbrace{\lambda}_{\ge 0}\underbrace{(Ax - b)}_{\le 0} + \underbrace{(1-\lambda)}_{\ge 0}\underbrace{(Ax' - b)}_{\le 0} \\ &\le 0. \end{aligned}$$

Hence $A[\lambda x + (1-\lambda)x'] \le b$, which is to say, $\lambda x + (1-\lambda)x'$ is feasible. □

5. Here is another example of a combinatorial optimization problem. We have not yet discussed methods for solving problems like this, but see if you can find a good approach. Your goal is to find the optimal solution to the problem and to *prove* that it is optimal.

> Indiana Jones has made it through the deadly traps of an ancient temple and has discovered ten treasures inside. Unfortunately, his knapsack is too small to carry them all, so he must choose (wisely). He has made the following estimates of the objects' weights and values. If his knapsack can hold at most 20 pounds of treasure, which objects should he take to maximize the value of his loot?

Treasure	Weight	Value
Crown of Atahualpa	4 lb.	$ 4,000
Itzcoatl's Orb	6 lb.	9,000
Tablet of the Heavens	7 lb.	7,000
Golden Quetzal	13 lb.	17,000
Mask of the Ancients	5 lb.	5,000

Treasure	Weight	Value
Key of Silver Light	2 lb.	$ 2,000
Idol of Inti	10 lb.	15,000
Eternal Quipu	5 lb.	8,000
Goblet of Uxmal	3 lb.	4,000
Sacred Stone of Cuzco	8 lb.	11,000

Solution. We might first try sorting the objects by value, and then choosing the highest-value items one by one, so long as we still have room in the knapsack. First we sort the treasures in descending order by value. (We'll put the Goblet of Uxmal ahead of the Crown of Atahualpa, because they are have the same value but the goblet weighs less.)

Treasure	Weight	Value
Golden Quetzal	13 lb.	$17,000
Idol of Inti	10 lb.	15,000
Sacred Stone of Cuzco	8 lb.	11,000
Itzcoatl's Orb	6 lb.	9,000
Eternal Quipu	5 lb.	8,000
Tablet of the Heavens	7 lb.	7,000
Mask of the Ancients	5 lb.	5,000
Goblet of Uxmal	3 lb.	4,000
Crown of Atahualpa	4 lb.	4,000
Key of Silver Light	2 lb.	2,000

Now let's fill the knapsack. We'll take the Golden Quetzal, since it has the highest value. Since the Golden Quetzal weighs 13 pounds, we have 7 pounds of capacity left. We cannot take either of the next two items in the list, the Idol of Inti or the Sacred Stone of Cuzco, because they weigh too much. So we'll take Itzcoatl's Orb, which weighs 6 pounds. This gives us 19 pounds, worth $26,000. Since there is no 1-pound item, this is the best we can do by this method. (Such a method, in which we simply take the "best" available choice at each step without considering how our choice will affect future options, is called a "greedy algorithm.")

But we can do better if we modify the method slightly. We weren't able to fill our knapsack because there is no 1-pound item to take. So let's try the same method, except that we will skip an item if it would give us a total of exactly 19 pounds. (We'll call this approach the "modified greedy algorithm.") As before we take the Golden Quetzal and skip the Idol of Inti and the Sacred Stone of Cuzco. This time we will also skip Itzcoatl's Orb and take the Eternal Quipu instead, giving us 18 pounds so far. Then we still have room to take the Key of Silver Light. This solution gives us a full 20 pounds of treasure worth $27,000.

However, there is another way to sort the treasure. Instead of sorting by value with no regard to weight, we can sort by value per pound. (As before, if two objects have the same value per pound, let's put the lighter one first.) Sorting the treasures in this way gives the following list.

Treasure	Weight	Value	Value per pound
Eternal Quipu	5 lb.	$ 8,000	$1,600/lb.
Itzcoatl's Orb	6 lb.	9,000	1,500/lb.
Idol of Inti	10 lb.	15,000	1,500/lb.
Sacred Stone of Cuzco	8 lb.	11,000	1,375/lb.
Goblet of Uxmal	3 lb.	4,000	1,333/lb.
Golden Quetzal	13 lb.	17,000	1,308/lb.
Key of Silver Light	2 lb.	2,000	1,000/lb.
Crown of Atahualpa	4 lb.	4,000	1,000/lb.
Mask of the Ancients	5 lb.	5,000	1,000/lb.
Tablet of the Heavens	7 lb.	7,000	1,000/lb.

If we apply the greedy algorithm to this list, we begin by taking the Eternal Quipu and Itzcoatl's Orb, bringing us to a total of 11 pounds. We do not have enough room for the Idol of Inti, but we can take the Sacred Stone of Cuzco. We must stop here, because we have reached 19 pounds; the value of our treasure is $28,000.

This is an improvement upon the solutions we obtained earlier. Perhaps we can improve even further if we try the modified greedy algorithm. We choose the Eternal Quipu and Itzcoatl's Orb and skip the Idol of Inti, as before, but now we skip the Sacred Stone of Cuzco as well, because that would give us a total of 19 pounds. Instead we take the Goblet of Uxmal, the Key of Silver Light, and the Crown of Atahualpa. We have filled our knapsack with 20 pounds, but when we calculate the value it turns out that we have done worse; we have only $27,000.

The greedy algorithm and the modified greedy algorithm are simple methods to use, and they usually work well to get an *approximation* of the best solution, but neither of these methods is *guaranteed* to give the best solution. In this problem there is a better solution than any of the four we have found so far.

Let's go about this in a very systematic way. We have already found a solution worth $28,000. If we are going to improve upon this, we must include in our solution at least one of the five highest-value items (the Golden Quetzal, the Idol of Inti, the Sacred Stone of Cuzco, Itzcoatl's Orb, or the Eternal Quipu) because the other five items are worth only $22,000 in all. So we will consider the five high-value items separately from the five low-value items.

The low-value items are listed below, sorted in ascending order by weight (and, incidentally, also by value).

Treasure	Weight	Value
Key of Silver Light	2 lb.	$2,000
Goblet of Uxmal	3 lb.	4,000
Crown of Atahualpa	4 lb.	4,000
Mask of the Ancients	5 lb.	5,000
Tablet of the Heavens	7 lb.	7,000

We will make a list of ways to form various weights from these low-value treasures. It is easy to see that the only way to make 2 pounds is to take the Key of Silver Light. Similarly, the only way to make 3 pounds is to take the Goblet of Uxmal, and the only way to make 4 pounds is to take the Crown of Atahualpa. To make 5 pounds, however, we have a choice: either we can take the Mask of the Ancients alone, or we can take the Key of Silver Light and the Goblet of Uxmal. In general, when our aim is to form a total weight of n pounds, we must also consider all of the possible ways to write n as a sum of smaller numbers (out of 2, 3, 4, 5, and 7). Following this approach, we construct the "low-value combinations" table on the following page.

To use this table, we will consider some combination of the high-value treasures, see how much room is left in the knapsack, and then find the best combination of low-value treasures in the table above that will fit. For example, if we choose the Sacred Stone of Cuzco and Itzcoatl's Orb as our high-value treasures, which together weigh 14 pounds, we will have 6 pounds of capacity remaining. Looking at the "6 lb." row and the rows above it in this table, we see that our best option is to take the Key of Silver Light and the Crown of Atahualpa (or, equivalently, the Key of Silver Light and the Goblet of Uxmal—perhaps this is actually a better choice, since it gives us the same value with less weight).

Proceeding in this way, we can find the best collection of treasures to take by considering all possible combinations of the high-value items; see the "high-value combinations" table on the next page. Of course, we can safely ignore those that put us overweight: the Golden Quetzal and the Idol of Inti together weigh 23 pounds, and the Golden Quetzal and the Sacred Stone of Cuzco together weigh 21 pounds. Furthermore, any combination of three or more high-value items will weigh more than 20 pounds (except the three lightest ones, the Sacred Stone of Cuzco, Itzcoatl's Orb, and the Eternal Quipu, which weigh 19 pounds together). These impossible combinations of high-value items are omitted from the table.

From these tables we see that the best choice is to take the Idol of Inti, the Eternal Quipu, the Key of Silver Light, and the Goblet of Uxmal, which weigh 20 pounds together and have a total value of $29,000. Moreover, this solution is unique.

(With the particular numbers used in this problem, it turns out that we would have found this optimal solution by using the greedy algorithm based on value per pound if we would have broken ties by putting the *heavier* item first rather than the lighter one. But this isn't a general rule that will always give the best possible solution.)

Low-value combinations

Weight	As a sum	Treasures	Value
2 lb.	2	Key	$ 2,000
3 lb.	3	Goblet	4,000
4 lb.	4	Crown	4,000
5 lb.	5	Mask	5,000
	$2+3$	Key, goblet	6,000
6 lb.	$2+4$	Key, crown	6,000
7 lb.	7	Tablet	7,000
	$2+5$	Key, mask	7,000
	$3+4$	Goblet, crown	8,000
8 lb.	$3+5$	Goblet, mask	9,000
9 lb.	$2+7$	Key, tablet	9,000
	$4+5$	Crown, mask	9,000
	$2+3+4$	Key, goblet, crown	10,000
10 lb.	$3+7$	Goblet, tablet	11,000
	$2+3+5$	Key, goblet, mask	11,000
11 lb.	$4+7$	Crown, tablet	11,000
	$2+4+5$	Key, crown, mask	11,000
12 lb.	$5+7$	Mask, tablet	12,000
	$2+3+7$	Key, goblet, tablet	13,000
	$3+4+5$	Goblet, crown, mask	13,000
13 lb.	$2+4+7$	Key, crown, tablet	13,000
14 lb.	$2+5+7$	Key, mask, tablet	14,000
	$3+4+7$	Goblet, crown, tablet	15,000
	$2+3+4+5$	Key, goblet, crown, mask	15,000
15 lb.	$3+5+7$	Goblet, mask, tablet	16,000
16 lb.	$4+5+7$	Crown, mask, tablet	16,000
	$2+3+4+7$	Key, goblet, crown, tablet	17,000
17 lb.	$2+3+5+7$	Key, goblet, mask, tablet	18,000
18 lb.	$2+4+5+7$	Key, crown, mask, tablet	18,000
19 lb.	$3+4+5+7$	Goblet, crown, mask, tablet	19,000
21 lb.	$2+3+4+5+7$	Key, goblet, crown, mask, tablet	22,000

High-value combinations

Quetzal	Idol	Stone	Orb	Quipu	Remaining	Low-value items	Total value
•					7 lb.	Goblet, crown	$25,000
	•				10 lb.	Goblet, tablet	26,000
		•			12 lb.	Key, goblet, tablet	24,000
			•		14 lb.	Goblet, crown, tablet	24,000
				•	15 lb.	Goblet, mask, tablet	24,000
•			•		1 lb.	*(None)*	26,000
•				•	2 lb.	Key	27,000
	•	•			2 lb.	Key	28,000
	•		•		4 lb.	Crown	28,000
	•			•	5 lb.	Key, goblet	29,000
		•	•		6 lb.	Key, crown	26,000
		•		•	7 lb.	Goblet, crown	27,000
			•	•	9 lb.	Key, goblet, crown	27,000
		•	•	•	1 lb.	*(None)*	28,000

Another approach is to formulate the problem as an *integer program*, which is like a linear program except that one or more variables are required to have integer values. We associate each treasure with a variable that is constrained to have the value 0 or 1; the value 1 will mean that the treasure is to be included in the knapsack, and the value 0 will mean that the treasure is to be left behind.

Treasure	Variable	Treasure	Variable
Crown of Atahualpa	c	Key of Silver Light	k
Itzcoatl's Orb	o	Idol of Inti	i
Tablet of the Heavens	t	Eternal Quipu	e
Golden Quetzal	q	Goblet of Uxmal	g
Mask of the Ancients	m	Sacred Stone of Cuzco	s

The following integer program maximizes the value of the items included in the knapsack, subject to the constraint that the total weight of the included items is less than 20 pounds, no item can be included more than once, and every item must be included an integer number of times. (The objective function is written in units of thousands of dollars to avoid a profusion of zeroes in the coefficients.)

$$
\begin{array}{lll}
\text{maximize} & 4c + 9o + 7t + 17q + 5m + 2k + 15i + 8e + 4g + 11s & \text{[value]} \\
\text{subject to} & 4c + 6o + 7t + 13q + 5m + 2k + 10i + 5e + 3g + 8s \le 20 & \text{[weight]} \\
& c \le 1 & \\
& o \le 1 & \\
& t \le 1 & \\
& q \le 1 & \\
& m \le 1 & \\
& k \le 1 & \\
& i \le 1 & \\
& e \le 1 & \\
& g \le 1 & \\
& s \le 1 &
\end{array}
$$

All variables nonnegative integers.

This integer program can be solved with Maple in the same way that a linear program is solved, the only change being `assume=nonnegint` in the *LPSolve* call rather than the usual `assume=nonnegative`. (Alternatively, using `assume=binary` constrains all variables to take values in the set $\{0, 1\}$, so if this option is used only the first constraint in the integer program is necessary.) Maple finds the optimal solution $c = 0$, $o = 0$, $t = 0$, $q = 0$, $m = 0$, $k = 1$, $i = 1$, $e = 1$, $g = 1$, $s = 0$, which corresponds to the solution we laboriously found above. □

6. The phrase "combinatorial optimization" is a mouthful. Come up with a shorter way to say it.

Solution. My first instinct was to shorten the phrase to "combo oppo," but I'm not sure how to justify the shortening of "optimization" to "oppo" except that it kinda sounds like a half-rhyme to "combo." "Combination" is often shortened to "combo," and "optimization" is sometimes shortened to "opti," so "combo opti" might be defensible. Something about it seems awkward, though. The straightforward abbreviation "C.O." is not too bad.

"Com op" (or the alternative spelling "comb op") was the most common submitted suggestion; this sounds pretty good. I also liked the suggestion "combi op." A couple of people suggested avoiding the long words altogether, preferring simpler, more straightforward descriptions like "searching for the best solution." The most aggressive suggestion was "combat optz," which is indeed a subsequence of "combinatorial optimization," but perhaps suggests a more violent field of study. □