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More examples of problems in NP.

Example. 2-COLORABILITY.

Instance: A simple undirected graph $G=(V,E)$.

Question: Is G 2-colorable? That is, does there exist a function $f: V \to \{1,2\}$ such that $f(u) \neq f(v)$ for all $\{u,v\} \in E$? (Equivalently, is G bipartite?)

Certificate: A function $f: V \to \{1,2\}$. This has size $O(|V|)$, which is polynomial in the size of the instance.

Verifier:

- For every $v \in V$, if $f(v) \notin \{1,2\}$, then output "no" and stop.
- For every $\{u,v\} \in E$, if $f(u) = f(v)$, then output "no" and stop.
- Output "yes."

This verification can be done in $O(|V|+|E|)$ time.

Example. 3-COLORABILITY.

[Just like 2-COLORABILITY.]

Example. HAMILTONIAN CIRCUIT.

Instance: A simple undirected graph $G=(V,E)$.

Question: Does G contain a Hamiltonian circuit (i.e., a spanning cycle)?

Certificate: A list of integers $(v_0, v_1, v_2, \ldots, v_n)$ naming the vertices in order around a Hamiltonian circuit. This has size $O(|V|)$.
— Or $O(|V| \log |V|)$, if you want to count bits.

Verifier:

- If $n \neq |V|$, output "no" and stop.
- If $v_0 \neq v_n$, output "no" and stop.
- For each $i \in \{0, 1, 2, \ldots, n-1\}$, if $v_i \notin \{1, 2, \ldots, n\}$, output "no" and stop.
- For each $i \in \{0, 1, 2, \ldots, n-2\}$, for each $j \in \{i+1, i+2, \ldots, n-1\}$, if $v_i = v_j$, output "no" and stop.
- For each $i \in \{0, 1, 2, \ldots, n-1\}$, if $\{v_i, v_{i+1}\} \notin E$, output "no" and stop.
- If $n < 3$, output "no" and stop.
- Output "yes."

This verification can be done in $O(|V|^2)$ time.

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Example. SAT. [P&S Example 15.7, §15.3]

Instance: A propositional formula F on the Boolean variables $x_1, \ldots, x_n$.

Question: Is F satisfiable? That is, does there exist an assignment of truth values to $x_1, \ldots, x_n$ such that the resulting truth value of F is TRUE?

Certificate: Truth values for all variables. This has size $O(n)$.

Verifier:

- If the certificate does not consist of exactly n bits, output "no" and stop.
- Evaluate F using the given truth values for the variables $x_1, \ldots, x_n$. If the result is FALSE, output "no" and stop.
- Output "yes."

This verification can be done in $O(m)$ time, where m is the length of the formula F. (Note: m is not the number of variables, because each variable may appear many times in F.)

Example. ILP. [P&S Example 15.8, §15.3]

Instance: An $m \times n$ matrix A of integers and a vector b of m integers.

Question: Does there exist a vector x of n integers such that $Ax = b$ and $x \geq 0$?

Certificate: A vector x of n integers.
[See P&S Example 15.8, and P&S Thm 13.4 in §13.3, for careful justification that a feasible IP always has a polynomial-size feasible solution.]

Verifier: Output "yes" iff all entries of x are integers, $Ax = b$, and $x \geq 0$.
This verification can be done with $O(mn)$ arithmetic operations.

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Aside: The class co-NP [P&S §16.1]

Defn. The complement of a decision problem A is the decision problem $\bar{A}$ in which an instance is the same as an instance of A and in which the answer to an instance x is "yes" if and only if the answer to x in A is "no."

Defn. The class co-NP is the class of decision problems whose complement is in NP.

- So, a decision problem is in co-NP iff all "no" instances have polynomial-size "co-certificates" proving that the answer is "no," verifiable by a "co-verifier" in polynomial time.
- Intuitively:
  - A decision problem is in NP when you can efficiently prove "yes" answers.
  - A decision problem is in co-NP when you can efficiently prove "no" answers.

Example. The complement of COMPOSITENESS is PRIMALITY (well, counting 1 as prime). COMPOSITENESS is in NP, so PRIMALITY is in co-NP.

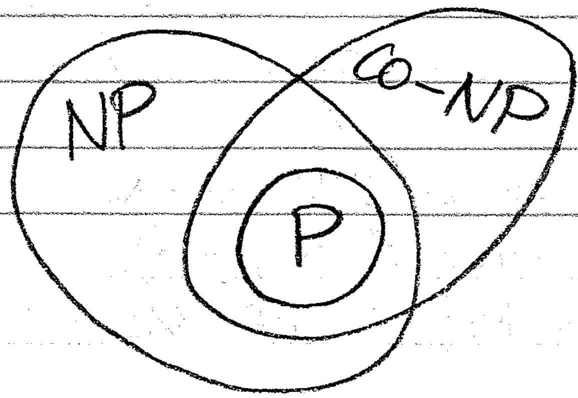
Example. 2-COLORABILITY is in co-NP, because a "co-certificate" to prove a "no" answer is an odd cycle. (A graph is bipartite if and only if it contains no odd cycle.)

Example. 3-COLORABILITY $\overset{?}{\notin}$ co-NP.
Nobody knows an efficient "co-certificate" to prove that a graph is not 3-colorable. (But, on the other hand, it is also true that nobody has proven 3-COLORABILITY is not in co-NP.)

Example. HAMILTONIAN CIRCUIT $\overset{?}{\notin}$ co-NP.
Same situation as for 3-COLORABILITY. Nobody knows an efficient way to prove that a general graph does not have a Hamiltonian circuit.

Example. P $\subseteq$ NP $\cap$ co-NP.
P $\subseteq$ co-NP for the same reason that P $\subseteq$ NP: the "co-certificate" can be nothing, and the "co-verifier" can verify a "no" answer by just solving the instance.

Open question: Is NP = co-NP? Conjecture: No.



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Polynomial-time reductions [P&S §15.4]

Defn. Let A_1 and A_2 be decision problems. We say that A_1 reduces in polynomial time to A_2 iff there exists a polynomial-time algorithm $\mathcal{A}_1$ for A_1 that uses a (hypothetical) algorithm $\mathcal{A}_2$ for A_2 as a subroutine at unit cost. We call $\mathcal{A}_1$ a polynomial-time reduction from A_1 to A_2.

Note: The phrase "at unit cost" in this definition means that in measuring the running time of $\mathcal{A}_1$ we are counting the execution of $\mathcal{A}_2$ as a single elementary operation.

In reality, of course, such an algorithm $\mathcal{A}_2$ for A_2 almost certainly takes many elementary operations. But counting $\mathcal{A}_2$ as a single elementary operation is justifiable in light of the following:

Proposition. [P&S Prop. 15.1]
If A_1 polynomially reduces to A_2 and there exists a polynomial-time algorithm for A_2, then there exists a polynomial-time algorithm for A_1.

Proof. Let the polynomial $p_1(n)$ bound the running time of $\mathcal{A}_1$ (with the assumption of unit-cost invocation of $\mathcal{A}_2$), and let the polynomial $p_2(n)$ bound the running time of $\mathcal{A}_2$. Then the actual number of elementary operations used to run $\mathcal{A}_1$ on an instance of size n, counting all operations used by the calls to $\mathcal{A}_2$, is bounded by

$$p(n) = p_1(n) \cdot p_2(p_1(n))$$

because $\mathcal{A}_1$ makes at most $p_1(n)$ calls to $\mathcal{A}_2$, and the largest possible input to $\mathcal{A}_2$ is $p_1(n)$ even if $\mathcal{A}_1$ used all of its steps just to write that input, so each call to $\mathcal{A}_2$ takes at most $p_2(p_1(n))$ elementary operations. Since $p(n)$ is a polynomial, this is a polynomial-time algorithm for A_1. $\square$

In a polynomial-time reduction, $\mathcal{A}_2$ may be called many times (well, only polynomially many times) by $\mathcal{A}_1$, and the operation of $\mathcal{A}_1$ may depend on the results of earlier calls to $\mathcal{A}_2$. But there is a particularly interesting kind of polynomial-time reduction in which $\mathcal{A}_1$ calls $\mathcal{A}_2$ only once, at the very end, and then directly returns the result from $\mathcal{A}_2$:

Poly-time reductions — 2

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<u>Defn.</u> We say that a decision problem A_1 <u>polynomially transforms</u> to another decision problem A_2 if there is a polynomial-time algorithm to convert any instance x of A_1 to an instance y of A_2 such that the answer to x is "yes" if and only if the answer to y is "yes".

<u>Example.</u> CNF-SAT polynomially transforms to ILP.

(CNF-SAT is a special case of SAT in which the instances are restricted to be formulas in conjunctive normal form.)

We saw an IP formulation for CNF-SAT in the lecture on June 24. For example, the CNF-SAT instance

$$(x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3) \wedge (x_2 \vee x_3 \vee \bar{x}_4)$$

can be converted (in polynomial time) to the IP

$$\begin{aligned}
\max\ & 0 \\
\text{s.t. } & x_1 + (1-x_2) + (1-x_3) \geq 1 \\
& (1-x_1) + x_3 \geq 1 \\
& x_2 + x_3 + (1-x_4) \geq 1 \\
& x_i \in \{0,1\} \text{ for all } i.
\end{aligned}$$

ILP is the decision version of integer programming; the question is, "Is this IP feasible? (Or, equivalently here, "Does this IP have a feasible solution with objective value ≥ 0?")

The answer to ILP for the IP formulation of a CNF-SAT instance is "yes" if and only if the original CNF formula is satisfiable. So this is a polynomial-time transformation. ✓

Example. HAMILTONIAN CIRCUIT polynomially transforms to TSP.

Recall the decision version of TSP:

Instance: n cities, the cost of each arc (i,j), and a value $L \in \mathbb{R}$.

Question: Does there exist a tour through all cities having total cost $\leq L$?

Given an instance of HAMILTONIAN CIRCUIT, i.e., a graph $G=(V,E)$, construct an instance of TSP as follows: set $n=|V|$, set the cost of arc (i,j) to 0 if $\{i,j\} \in E$ or 1 otherwise, set $L=0$. Then the answer to the TSP instance is "yes" (i.e., there exists a tour of total cost 0) if and only if G has a Hamiltonian circuit. This conversion can be done in polynomial time, so this is a polynomial-time transformation. ✓

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<u>Example.</u> CLIQUE polynomially transforms to INDEPENDENT SET.

CLIQUE:

Instance: Graph $G=(V,E)$, integer k.

Question: Does G contain a clique of size k, i.e., a subset $K \subseteq V$ with $|K|=k$ such that every two vertices in K are adjacent?

INDEPENDENT SET:

Instance: Graph $G=(V,E)$, integer k.

Question: Does G contain an independent set of size k, i.e., a subset $S \subseteq V$ with $|S|=k$ such that no two vertices in S are adjacent?

Given an instance (G,k) of CLIQUE, convert G to its complement $\overline{G}$ (change edges to non-edges and vice versa) to get an instance $(\overline{G},k)$ of INDEPENDENT SET. The graph $\overline{G}$ has an independent set of size k if and only if G has a clique of size k. ✓

<u>Defn.</u> A decision problem A is called <u>NP-complete</u> if

- $A \in NP$ and
- all other problems in NP polynomially transform to A.

At the moment it is not clear that any such problems exist (this is the result of Cook's theorem — tomorrow's lecture), but

- <u>if</u> a decision problem A is NP-complete, and

- <u>if</u> there exists a polynomial-time algorithm for A,

then, as a consequence of the proposition from earlier, we would have a polynomial-time algorithm for <u>all</u> problems in NP!

This would mean P=NP, which <u>appears</u> not to be true (because no one has ever been successful in finding a polynomial-time algorithm for any NP-complete problem).

So, in a meaningful sense, NP-complete problems are the <u>hardest</u> problems in NP: if we could solve any NP-complete problem in poly-time, then we could solve <u>all</u> problems in NP in poly time.