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Some hard problems (and their IP formulations)

All of the problems that we will discuss today are NP-hard (more specifically, all of the decision problems, and the decision versions of the other problems, are NP-complete). We will learn what that means next week, but in a nutshell it means that no one has ever been able to find algorithms to solve these problems that are always efficient and always correct.

Hard problems we have already seen:

- Knapsack problem
- Boolean satisfiability (SAT) problem
- Integer programming in general

Traveling salesman problem (TSP)

Input:
- Positive integer n, number of cities.
- $n \times n$ matrix $[c_{ij}]$ giving costs of traveling from city i to city j.
  — Not necessarily symmetric.

Objective: Find a (directed) tour through the cities that visits each city exactly once and has minimum total cost.

IP formulation.

Variables $x_{ij} \in \{0,1\}$ for all $i \neq j$.

$x_{ij} = 1$ iff city i is followed by city j in the tour.

$$\min \sum_i \sum_{j \neq i} c_{ij} x_{ij}$$

$$\text{s.t.} \sum_{j \neq i} x_{ij} = 1 \quad \text{for all } i$$

[must leave each city exactly once]

$$\sum_{i \neq j} x_{ij} = 1 \quad \text{for all } j$$

[must enter each city exactly once]

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1 \quad \text{for all } \emptyset \subsetneq S \subsetneq \{1,2,\ldots,n\}$$

[subtour elimination]

—see next page

$$x_{ij} \in \{0,1\} \quad \text{for all } i,j$$

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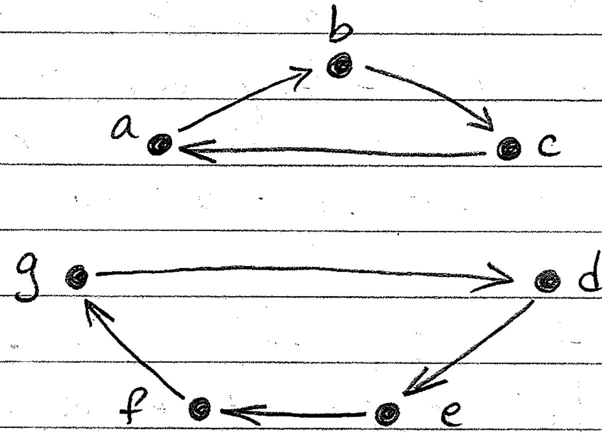
Meaning of the subtour elimination constraints:
Without these constraints, invalid sets of links such as



would be feasible (because this set of links leaves each city exactly once and enters each city exactly once), even though the links do not form a single tour that passes through all cities.

The subtour elimination constraints enforce the requirement that for every nonempty proper subset S of cities, there must be at least one link <u>leaving</u> S. This disallows invalid sets of links such as the one above, which has no link leaving $S = \{a, b, c\}$ (and also no link leaving $S = \{d, e, f, g\}$).

- Unfortunately this formulation has $2^n - 2$ subtour elimination constraints, which quickly becomes impractical as n increases. See Example 13.1 in P&S §13.1 for an alternative formulation with fewer constraints.

Hamiltonian circuit

Input: • Simple undirected graph $G=(V,E)$.

Objective: Find a spanning cycle, i.e., a cycle that passes through every vertex. (A spanning cycle is called a Hamiltonian circuit.)
— Or report that no Hamiltonian circuit exists.

IP formulation

Variables: $x_e \in \{0,1\}$ for each $e \in E$, indicating whether or not e is included in the Hamiltonian circuit.

$$\max \quad 0$$

[not optimizing, just searching for feasible solution]

$$\text{s.t.} \sum_{\substack{e \in E \text{ such that} \\ e \text{ is incident} \\ \text{upon } v}} x_e = 2 \quad \text{for all } v \in V$$

[every vertex must have degree 2]

$$\sum_{i \in S} \sum_{j \notin S} x_{\{i,j\}} \geq 1 \quad \text{for all } \emptyset \subsetneq S \subsetneq V$$

[subtour elimination]

$$x_e \in \{0,1\} \text{ for all } e \in E.$$

Or: This problem can be reduced to TSP by assigning each edge the cost 0 and each non-edge the cost 1 and finding a min-cost tour. If the cost of this tour is 0, it's a Hamiltonian circuit.

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Contrast the Hamiltonian circuit problem with the Eulerian circuit problem:

Input: An undirected graph $G=(V,E)$.

Objective: Find a closed walk that uses every edge exactly once. (Such a walk is called an Eulerian circuit.)
— Or report that no Eulerian circuit exists.

Decision version: Does there exist an Eulerian circuit in G? (Yes or no)

Euler proved in 1736 what is often considered the first theorem in graph theory:

Theorem. A graph has an Eulerian circuit if and only if it is connected and the degree of every vertex is even.

So the decision version of the Eulerian circuit problem is very easy to solve. (Finding an Eulerian circuit if one exists is also easy.)

In contrast, no one knows an efficient method for solving the Hamiltonian circuit problem — that problem seems to be inherently difficult.

Independent set

Input: A simple undirected graph $G=(V,E)$.

Objective: Find a set $S \subseteq V$ of maximum cardinality so that no two vertices in S are adjacent. (Such a set S is called an independent set or a stable set.)

IP formulation:

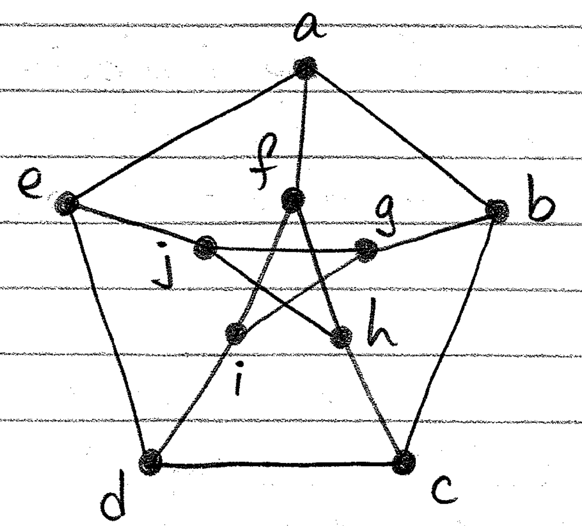
Variables: $x_v \in \{0,1\}$ for each $v \in V$, indicating whether $v \in S$.

$$\max \sum_{v \in V} x_v$$

$$\text{s.t. } x_i + x_j \leq 1 \text{ for all } \{i,j\} \in E$$

$$x_v \in \{0,1\} \text{ for all } v \in V.$$

Example: The Petersen graph.



Maximum independent set has cardinality 4, e.g., $\{a, c, i, j\}$.

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Max clique

Input: A simple undirected graph $G=(V,E)$.

Objective: Find a set $S \subseteq V$ of maximum cardinality so that every two vertices in S are adjacent. (Such a set S is called a clique.)

IP formulation:

Variables: $x_v \in \{0,1\}$ for each $v \in V$, indicating whether $v \in S$.

$$\max \sum_{v \in V} x_v$$

$$\text{s.t. } x_i + x_j \le 1 \text{ for all } \{i,j\} \notin E$$

$$x_v \in \{0,1\} \text{ for all } v \in V.$$

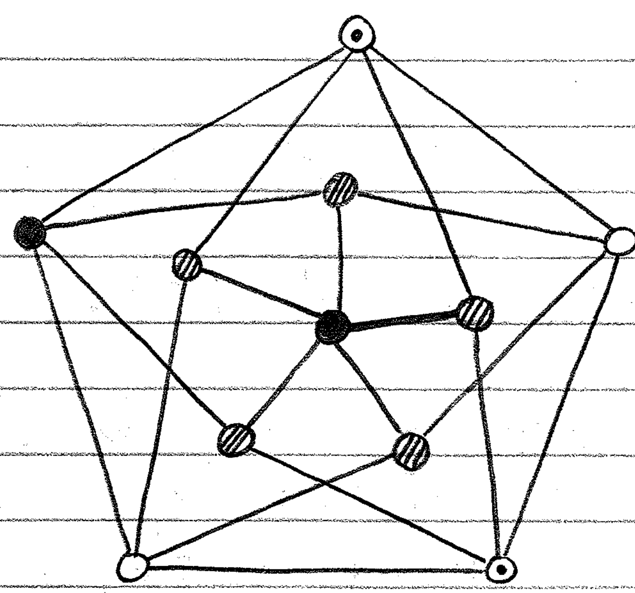
Or: This problem can be reduced to the independent set problem: Find the largest independent set in the complement of G (complement: replace all edges with non-edges and vice versa).

Chromatic number

Input: A simple undirected graph $G=(V,E)$.

Objective: Find the minimum positive integer χ such that G has a proper (vertex) χ-coloring: a function $f: V \to \{1, 2, \ldots, \chi\}$ such that $f(i) \neq f(j)$ for every edge $\{i,j\} \in E$. (Think of $\{1, 2, \ldots, \chi\}$ as colors assigned to the vertices; adjacent vertices must be assigned different colors.) This minimum χ is the *chromatic number* of G.

Example: The Grötzsch graph.



This is a proper 4-coloring with the "colors" ○, ●, ⊙, and ⊘.

- It's pretty easy to see there is no proper 2-coloring.
- Exercise: There is no proper 3-coloring.

So the chromatic number of this graph is 4.

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Chromatic number — 2

IP formulation: Let $n = |V|$.

Variables: • $x_{vk} \in \{0,1\}$ for $v \in V$, $k \in \{1,2,\ldots,n\}$ indicating whether vertex v is colored with color k.

— Note: We don't know ahead of time how many colors are needed, but certainly n colors are enough.

• $u_k \in \{0,1\}$ for $k \in \{1,2,\ldots,n\}$ indicating whether color k is used anywhere.

$$\min \sum_{k=1}^{n} u_k \qquad \text{[number of colors used]}$$

s.t. $x_{ik} + x_{jk} \leq 1$ for all $\{i,j\} \in E$, $k \in \{1,2,\ldots,n\}$

[adjacent vertices can't have same color]

$x_{vk} \leq u_k$ for all $v \in V$, $k \in \{1,2,\ldots,n\}$

[if v has color k then $u_k = 1$]

$\sum_{k=1}^{n} x_{vk} = 1$ for all $v \in V$ [every vertex gets a color]

$x_{vk} \in \{0,1\}$ for all v, k

$u_k \in \{0,1\}$ for all k.

Minimum vertex cover

Input: A simple undirected graph $G=(V,E)$.

Objective: Find a subset $S \subseteq V$ of minimum cardinality such that for every edge $\{i,j\} \in E$ at least one of the endpoints i and j is in S.

IP formulation

Variables: $x_v \in \{0,1\}$ for each $v \in V$ indicating whether $v \in S$.

$$\min \sum_{v \in V} x_v$$

$$\text{s.t.} \quad x_i + x_j \geq 1 \quad \text{for every } \{i,j\} \in E$$

$$x_v \in \{0,1\} \quad \text{for all } v \in V$$

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Minimum dominating set

Input: A simple undirected graph $G=(V,E)$

Objective: Find a subset $S \subseteq V$ of minimum cardinality such that for every vertex $v \in V$, either $v \in S$ or $\{u,v\} \in E$ for some $u \in S$. (In other words, every vertex is either in S or adjacent to a vertex in S.)
— Such a set S is called a dominating set.

IP formulation

Variables: $x_v \in \{0,1\}$ for each $v \in V$ indicating whether $v \in S$.

$$\min \sum_{v \in V} x_v$$

$$\text{s.t. } x_v + \sum_{\substack{u \in V \\ \text{such that} \\ \{u,v\} \in E}} x_u \geq 1 \quad \text{for every } v \in V$$

$$x_v \in \{0,1\} \quad \text{for all } v \in V$$

Subset sum

Input: A list $(c_1, c_2, \dots, c_n)$ of integers and an integer s.

Question: Does there exist a subset $I \subseteq \{1, 2, \dots, n\}$ such that $\sum_{i \in I} c_i = s$?

IP formulation

Variables: x_i for $1 \le i \le n$, indicating whether $i \in I$.

$$\max \ 0$$

$$\text{s.t.} \ \sum_{i=1}^{n} c_i x_i = s$$

$$x_i \in \{0, 1\} \text{ for all } i$$

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Bin packing

Input: A list $(c_1, c_2, \ldots, c_n)$ of positive integers (item sizes) and a positive integer bin capacity B.

Objective: Determine the minimum number of bins of capacity B needed to hold all n items without exceeding the capacity of any bin.

IP formulation

Variables:
- $x_{ij} \in \{0,1\}$ for $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$ indicating whether item i is to be placed in bin j.
- $u_j \in \{0,1\}$ for $j \in \{1, \ldots, n\}$ indicating whether bin j is used.

— Note: We don't know ahead of time how many bins are necessary, but n bins will be enough, unless $c_i > B$ for some i (but then it's infeasible anyway).

(Bin packing)

$$\min \sum_{j=1}^{n} u_j \quad \text{[number of bins]}$$

$$\text{s.t.} \sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i \quad \text{[every item must be packed]}$$

$$\sum_{i=1}^{n} c_i x_{ij} \leq B \text{ for all } j \quad \text{[cannot exceed bin capacity]}$$

$$x_{ij} \leq u_j \text{ for all } i,j \quad \text{[if } x_{ij} = 1 \text{ then } u_j = 1\text{]}$$

$$x_{ij} \in \{0,1\} \text{ for all } i,j$$

$$u_j \in \{0,1\} \text{ for all } j$$

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BETWEENNESS

Input:
- Finite set A of cardinality n
- Collection C of ordered triples (a,b,c) of distinct elements of A

Question: Does there exist a permutation of A in which, for every $(a,b,c) \in C$, b is between a and c?

– That is: Does there exist a bijection $f: A \to \{1, 2, \ldots, n\}$ such that for all $(a,b,c) \in C$ either $f(a) < f(b) < f(c)$ or $f(c) < f(b) < f(a)$?

IP formulation

Variables:

- $x_{ij} \in \{0,1\}$ for all $i,j \in A$, $i \neq j$, indicating whether $f(i) < f(j)$.

- $y_{abc} \in \{0,1\}$ for all $(a,b,c) \in C$, indicating whether $f(a) < f(b) < f(c)$.

[So $x_{ij} = 0$ means $f(i) > f(j)$
and $y_{abc} = 0$ means $f(c) < f(b) < f(a)$.]

(BETWEENNESS)

$$\max\ 0$$

s.t. $x_{ij} + x_{ji} = 1$ for all $i \neq j$ [antisymmetry]

$x_{ij} + x_{jk} \leq 1 + x_{ik}$ for all distinct i, j, k
[transitivity: if $x_{ij} = 1$ and $x_{jk} = 1$ then $x_{ik} = 1$]

$x_{ab} + x_{bc} \geq 2\, y_{abc}$ for all $(a,b,c) \in C$
[if $y_{abc} = 1$, then $f(a) < f(b) < f(c)$]

$x_{cb} + x_{ba} \geq 2(1 - y_{abc})$ for all $(a,b,c) \in C$
[if $y_{abc} = 0$, then $f(c) < f(b) < f(a)$]

$x_{ij} \in \{0,1\}$ for all $i \neq j$

$y_{abc} \in \{0,1\}$ for all $(a,b,c) \in C$

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Minimum set cover

Input:
- Finite set S
- Collection $\mathcal{C}$ of subsets of S

Objective: Find a subset $\mathcal{C}' \subseteq \mathcal{C}$ of minimum cardinality such that $S \subseteq \bigcup \mathcal{C}'$ (that is, every element in S is in at least one subset in $\mathcal{C}'$). — Or determine that $S \nsubseteq \bigcup \mathcal{C}$.

IP formulation:

Variables: $x_C \in \{0,1\}$ for every $C \in \mathcal{C}$ indicating whether $C \in \mathcal{C}'$.

$$\min \sum_{C \in \mathcal{C}} x_C$$

$$\text{s.t.} \sum_{\substack{C \in \mathcal{C} \\ \text{such that} \\ s \in C}} x_C \geq 1 \quad \text{for all } s \in S$$

[every $s \in S$ must be included in some $C \in \mathcal{C}'$]

$$x_C \in \{0,1\} \quad \text{for all } C \in \mathcal{C}$$