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The primal-dual method applied to max flow [P&S §5.6]

Recall the primal-dual algorithm. It involves four LPs:

PRIMAL

$$\min\ c^T x$$
$$\text{s.t. } Ax = b \geq 0$$
$$x \geq 0$$

DUAL

$$\max\ b^T \pi$$
$$\text{s.t. } A^T \pi \leq c^T$$
$$\pi \text{ unrestricted}$$

Admissible set (of primal variables or dual constraints):

$$J = \{ j : \text{jth dual constraint is tight} \}$$
$$[\text{i.e., } (A_j)^T \pi = c_j]$$

RESTRICTED PRIMAL

$$\min\ \xi = \sum_{i=1}^{m} r_i$$

$$\text{s.t. } \sum_{j \in J} a_{ij} x_j + r_i = b_i \text{ for all } i$$
$$x_j \geq 0 \text{ for all } j \in J$$
$$x_j = 0 \text{ for all } j \notin J$$
$$r_i \geq 0 \text{ for all } i$$

DUAL OF RESTRICTED PRIMAL (DRP)

$$\max\ b^T \pi$$

$$\text{s.t. } (A_j)^T \pi \leq 0 \text{ for } j \in J$$
$$\pi_i \leq 1 \text{ for all } i$$
$$\pi \text{ unrestricted.}$$

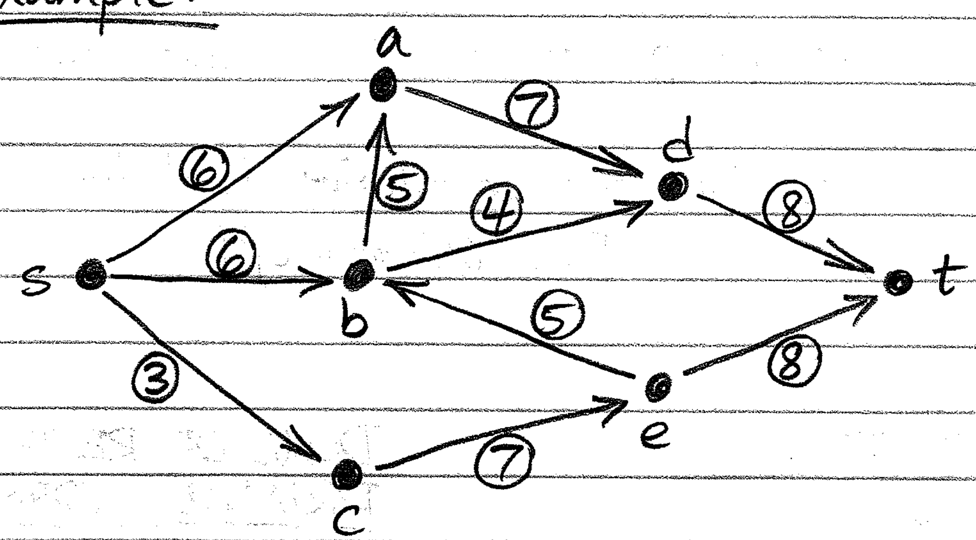
The max-flow problem [see also P&S §4.3]

Given: • A directed graph $G=(V,E)$;
• Specified nodes s and t;
• Arc capacities $b_{ij} > 0$ for all arcs $(i,j) \in E$.

[This information is called a flow network $N=(s,t,V,E,b)$.]

Goal: Maximize the total flow from s to t subject to the arc capacities.

Example:



Arc capacities circled.

One feasible flow in this network has the following flows along the arcs:
6 along $s \to a$, 2 along $s \to c$, 6 along $a \to d$,
2 along $b \to d$, 2 along $c \to e$, 8 along $d \to t$,
2 along $e \to b$, zero along all other arcs.

In this flow, the net outflow at s is 8 (and the net inflow at t is 8), so the value of this flow is 8. Total inflow equals total outflow at each other node (flow conservation).

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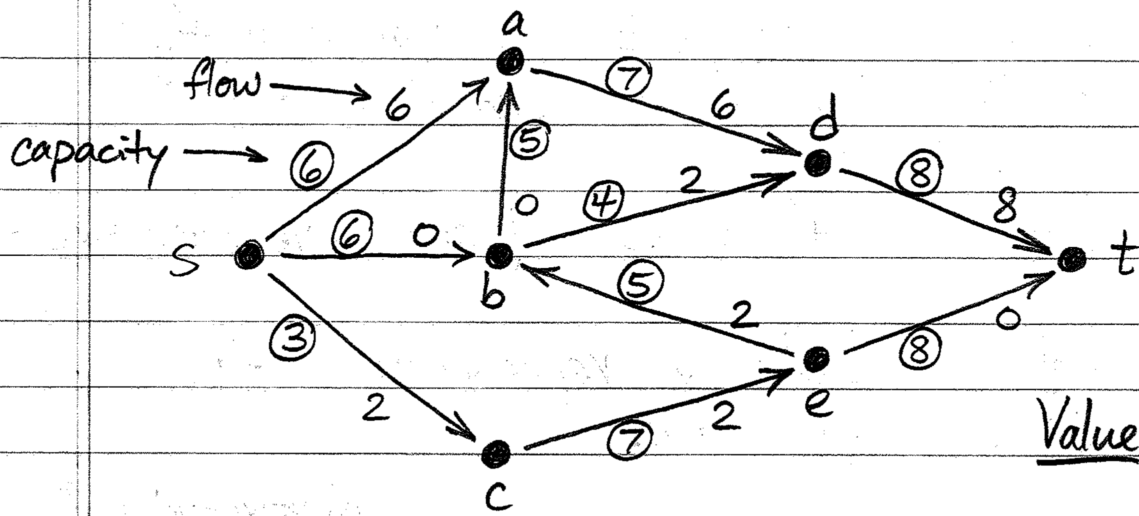
Max-flow problem — 2

Illustration of this feasible flow:



Value of flow is 8.

Note that this particular feasible flow is not a maximum feasible flow, which is what we seek in the max-flow problem.

(The max flow in this example has value 11.)

Max-flow: LP formulation.

Variables and domains:

- For each arc $(i,j) \in E$, a variable f_{ij} representing the flow along the arc.

  Domain: $f_{ij} \geq 0$ [can't have backward flow]

- A single variable v representing the value of the flow.

  Domain: v unrestricted.

[Negative-value flows are possible: for example,

has value -3.]

Objective: Maximize v.

Constraints: Let A be the node-arc incidence matrix of G (a row for each node, a column for each arc; entry a_{ij} is $+1$ if arc e_j leaves node i, or -1 if arc e_j enters node i, or 0 otherwise). [see P&S page 75]
Then, if a_i is the ith row of A, $a_i f$ is the net outflow at node i. We want this to be $+v$ at node s, $-v$ at row t, and 0 elsewhere:

$$Af = \begin{array}{c@{\;}l} \begin{bmatrix} +v \\ -v \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \leftarrow \text{row } s \\ \leftarrow \text{row } t \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{all other rows} \end{array} \end{array}$$

[flow-balance constraints]

Max-flow LP formulation — ②

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We also need to respect the arc capacities:

$$f \leq b \qquad \text{[capacity constraints]}.$$

So the LP formulation for max-flow is

$$\begin{aligned}
\max \quad & v \\
\text{s.t.} \quad & Af = \begin{bmatrix} +v \\ -v \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{row } s \\ \leftarrow \text{row } t \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{all other rows} \end{array} \qquad \text{[flow balance]} \\
& f \leq b \qquad \text{[capacities]} \\
& f \geq 0, \ v \text{ unrestricted.}
\end{aligned}$$

We want to apply the primal-dual algorithm. This is a max LP, so we will view it as the DUAL in the primal-dual framework. In order to do this, we would like this LP to be in the form of the dual recalled earlier, i.e., all $\leq$ constraints and all variables unrestricted.

Define

$$d = \begin{bmatrix} -1 \\ +1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{row } s \\ \leftarrow \text{row } t \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{all other rows} \end{array}$$

(Note sign flip from RHS of flow-balance constraints.)

Then we can rewrite the max-flow LP as

$$(D)\begin{cases} \max & v \\ \text{s.t.} & Af + d^T v \leq 0 \quad \text{[flow balance]} \\ & f \leq b \quad \text{[capacities]} \\ & -f \leq 0 \quad \text{[nonnegative flow]} \\ & f \text{ unrestricted, } v \text{ unrestricted.} \end{cases}$$

Notes:

- The reason for the sign flip in d is because $d^T v$ was moved to the LHS of the constraint, in order to make the RHS constant.
- Flow balance has been rewritten with $\leq$ instead of $=$. This is OK because if there is a deficit in flow balance at some node then there must be a surplus in flow balance at some other node, so the only way all the $\leq$ constraints can be satisfied is with equality at every node.
  - Consequently the flow-balance constraints will always be tight for any feasible flow. The only reason we wrote them as $\leq$ constraints is to fit the form of the dual in the primal-dual framework.
- The domains $f \geq 0$ have been reinterpreted as constraints $-f \leq 0$, because we want unrestricted variables in the dual in this framework.

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Dual of restricted primal for max flow

Observe how we form the DRP from the dual in the primal–dual framework:
- Replace RHS of dual constraints with 0.
- Delete constraints not in J (i.e., delete constraints that are not tight in the dual).
- Add constraints requiring all variables to be ≤ 1.

So for max-flow the DRP is:

[because flow-balance constraints are always tight in (D)]

$$
(\text{DRP}) \left\{
\begin{array}{lll}
\max & v & \\
\text{s.t.} & Af + d^T v \leq 0, & \text{for all nodes} \\
& f_{ij} \leq 0, & \text{for arcs } (i,j) \text{ where } f_{ij} = b_{ij} \text{ in } (D) \\
& -f_{ij} \leq 0, & \text{for arcs } (i,j) \text{ where } f_{ij} = 0 \text{ in } (D) \\
& f_{ij} \leq 1, & \text{for all arcs } (i,j) \\
& v \leq 1 & \\
& f \text{ unrestricted, } v \text{ unrestricted.} &
\end{array}
\right.
$$

<u>Interpretation</u>: Attempt to find an $s \to t$ flow of value $v = 1$ such that
- arcs (i,j) <u>saturated</u> in the dual ($f_{ij} = b_{ij}$) can have <u>backward</u> flow in DRP ($f_{ij} \leq 0$);
- arcs (i,j) with <u>zero</u> flow in the dual ($f_{ij} = 0$) can have <u>forward</u> flow in DRP ($-f_{ij} \leq 0$);
- other arcs can have flow in <u>either</u> direction in DRP.

[Note that the arc capacities are totally absent from the DRP. Finding an optimal solution to the DRP boils down to a reachability problem (as it did also for shortest path). We have removed consideration of numerical values from the problem and have reduced it to a consideration of structural properties only — we have "combinatorialized" the RHS of the original LP. See P&S §5.5 for a deeper discussion of this idea.]

An $s \to t$ flow of value 1 that is an optimal solution to the DRP is called an augmenting path.

To find an augmenting path given a feasible flow f, we construct an auxiliary network:

- For every arc (i,j) that is saturated by f ($f_{ij} = b_{ij}$), the corresponding arc in the auxiliary network points backward (from j to i).
- For every arc (i,j) that has zero flow in f ($f_{ij} = 0$), the corresponding arc in the auxiliary network points forward (from i to j).
- For every other arc (i,j), the corresponding arc in the auxiliary network points in both directions (or it is a pair of arcs, one in each direction).

DRP for max flow, augmenting paths — 2

Once we have the auxiliary network, then we just need to find a (directed) path from s to t in it — this will be an augmenting path, giving an optimal solution to the DRP.

When we have an optimal solution to the DRP, we use it to improve the dual solution by adding a multiple of the optimal DRP solution to the current dual solution — as large a multiple as possible while keeping the dual solution feasible.
- This is directly from the primal-dual algorithm.

What does it mean to add a multiple of the optimal DRP solution to the current dual solution? It means to <u>increase</u> flow along forward arcs of the augmenting path and to <u>decrease</u> flow along backward arcs.
- This is why saturated arcs must go backward: their flow cannot be increased. And arcs with zero flow must go forward: their flow cannot be decreased. Other arcs can go in either direction: their flow can be increased or decreased.

How much can the flow along an arc be changed?

- If the augmenting path traverses the arc in the forward direction, then its flow will be increased, so the maximum change is $b_{ij} - f_{ij}$ (i.e., remaining capacity).

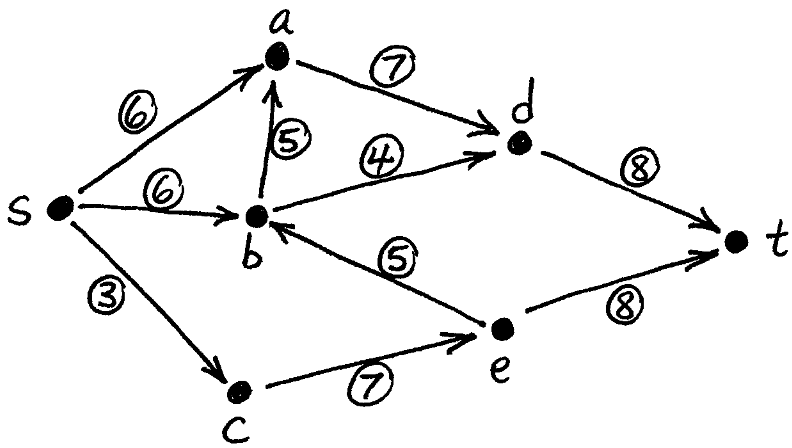
- If the augmenting path traverses the arc in the backward direction, then its flow will be decreased, so the maximum change is f_{ij} (can't decrease past zero).

So the maximum change of flow along an augmenting path P is

$$\min_{\text{arcs of } P} \left\{ \begin{array}{ll} b_{ij} - f_{ij} & \text{along forward arcs;} \\ f_{ij} & \text{along backward arcs} \end{array} \right\}.$$

If the auxiliary network has no directed path from s to t, then there is no augmenting path, the optimal objective value for DRP is 0, so the optimal objective value for the restricted primal is $\xi_{opt} = 0$, so the current dual solution is optimal.

18 June. <u>Augmenting paths for max flow — Example.</u>



(Arc capacities circled.)

Primal-dual method begins with a feasible solution to the dual, which in this case is a feasible flow. Let's begin with the all-zero flow — certainly that's feasible.

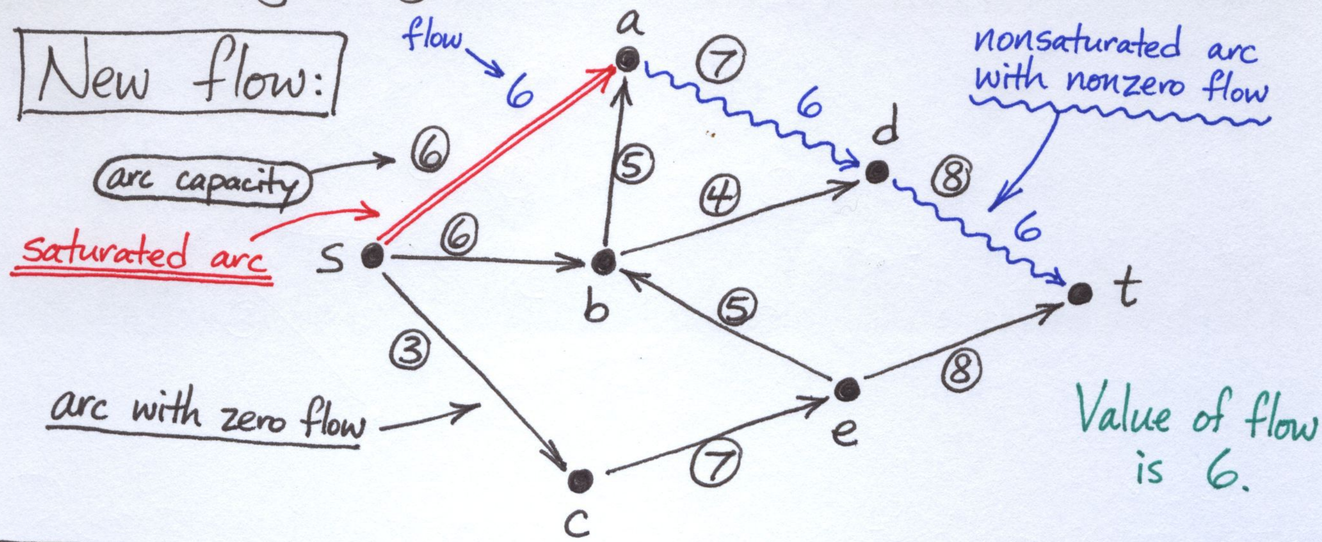
All arcs have zero flow, so every arc in the auxiliary network points <u>forward</u>. So, at the beginning, the auxiliary network is the same as the flow network (but arc capacities are irrelevant). We find a directed $s \to t$ path in the auxiliary network, say $s - a - d - t$. This is our augmenting path. All these arcs (in the flow network) are traversed in the forward direction by the augmenting path, so we will be <u>increasing</u> the flow along each arc.

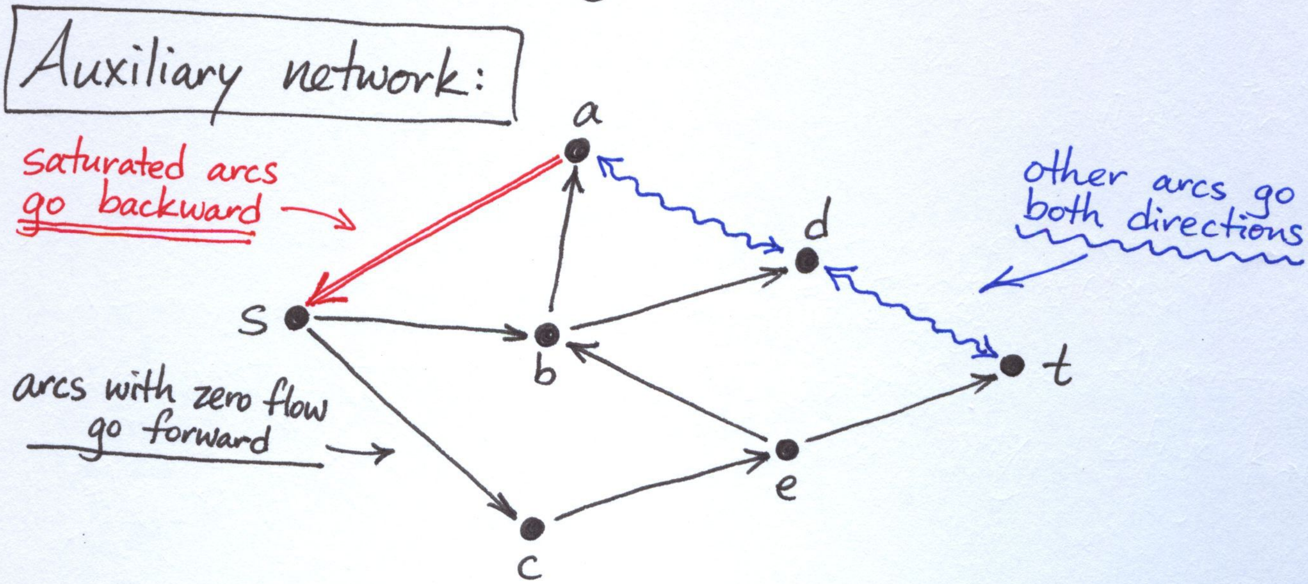
Maximum changes are:

$$\left.\begin{array}{l} \bullet \text{ for } (s,a),\ b_{sa} - f_{sa} = 6 - 0 = 6 \\ \bullet \text{ for } (a,d),\ b_{ad} - f_{ad} = 7 - 0 = 7 \\ \bullet \text{ for } (d,t),\ b_{dt} - f_{dt} = 8 - 0 = 8 \end{array}\right\} \text{minimum} = 6.$$

So <u>increase</u> the flows along these arcs by 6 to improve the dual solution.

18 June. Augmenting paths for max flow — Example — 2





Augmenting path is a directed $s \to t$ path in the auxiliary network, say s–c–e–b–d–t.
Maximum changes in flow along these arcs:

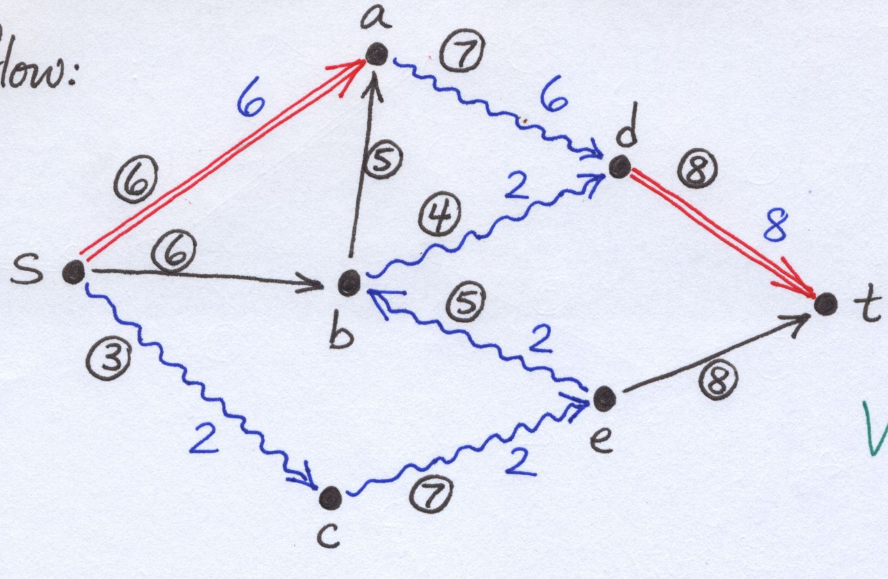
- (s,c), forward arc, max increase $b_{sc} - f_{sc} = 3-0 = 3$.
- (c,e), forward arc, max increase $b_{ce} - f_{ce} = 7-0 = 7$.
- (e,b), forward arc, max increase $b_{eb} - f_{eb} = 5-0 = 5$.
- (b,d), forward arc, max increase $b_{bd} - f_{bd} = 4-0 = 4$.
- (d,t), forward arc, max increase $b_{dt} - f_{dt} = 8-6 = 2$.

minimum = 2.

So change flow along these arcs by 2.

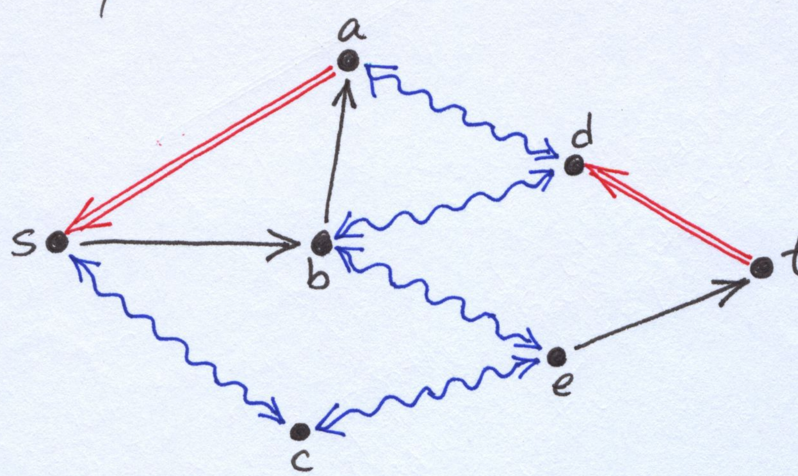
18 June. Augmenting paths for max flow — Example — ③

· New flow:



Value of flow is 8.

Auxiliary network:



Augmenting path: Let's take s — b — e — t.

Maximum changes in flow along these arcs:

- (s,b), forward arc, max increase $b_{sb} - f_{sb} = 6 - 0 = 6$.
- (b,e), <u>backward</u> arc, max <u>decrease</u> $f_{eb} = 2$.
- (e,t), forward arc, max increase $b_{et} - f_{et} = 8 - 0 = 8$.

min = 2.

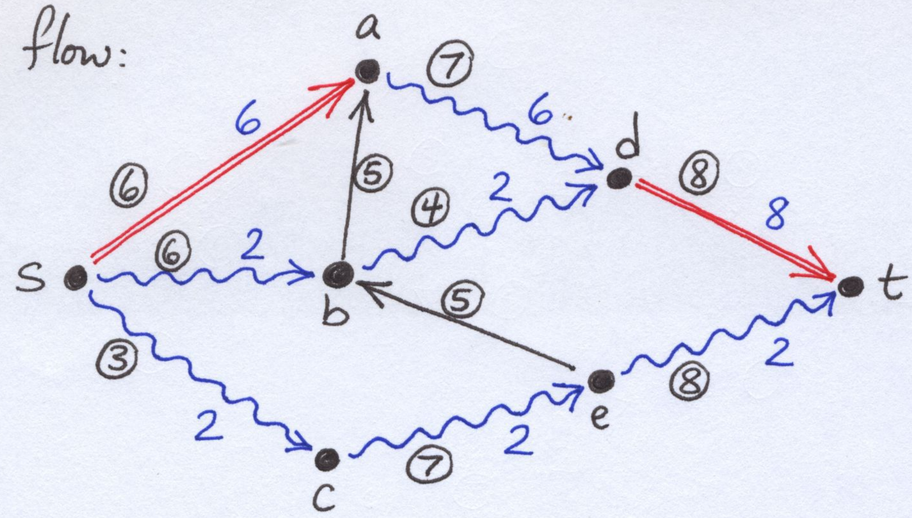
So adjust flows along this path by 2.

— <u>Increase</u> flow by 2 along (s,b) and (e,t).

— <u>Decrease</u> flow by 2 along (e,b).

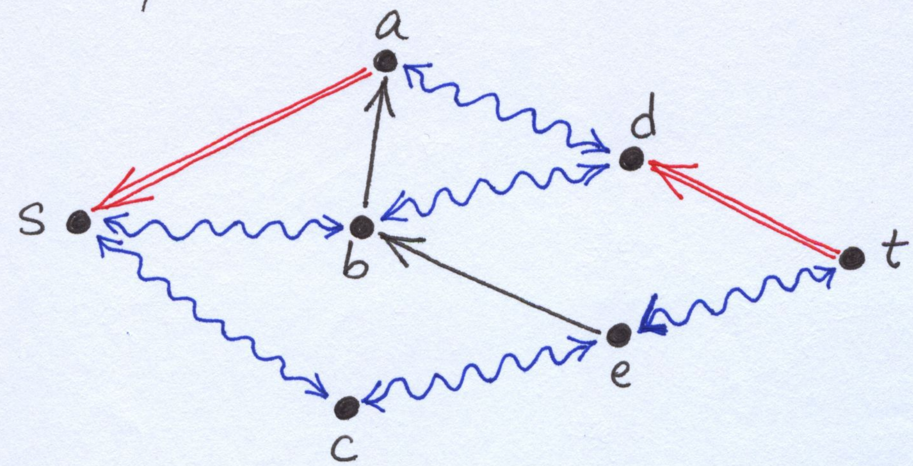
18 June. Augmenting paths for max flow — Example — 4

New flow:



Value of flow is 10.

Auxiliary network:



Augmenting path: s — c — e — t.

Maximum changes in flow along these arcs:

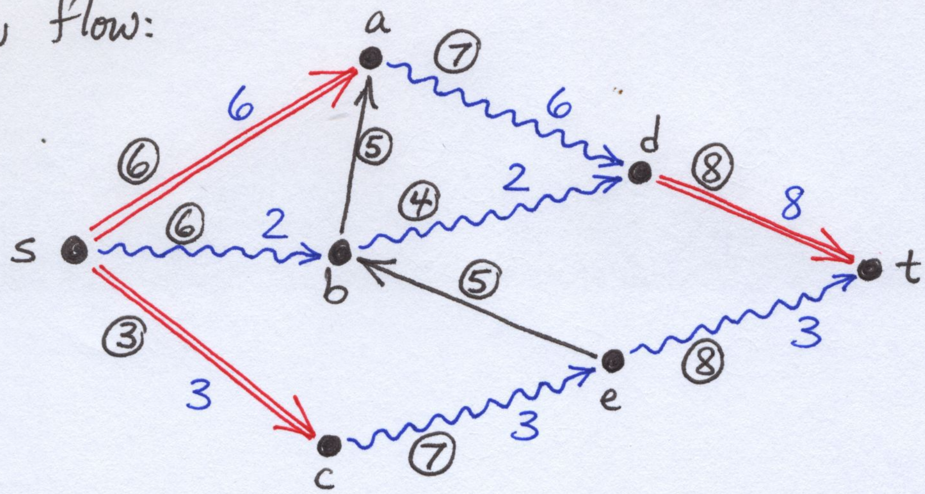
- (s,c), forward arc, max increase $b_{sc} - f_{sc} = 3-2 = 1$.
- (c,e), forward arc, max increase $b_{ce} - f_{ce} = 7-2 = 5$.
- (e,t), forward arc, max increase $b_{et} - f_{et} = 8-2 = 6$.

min = 1.

So adjust flows along this path by 1.

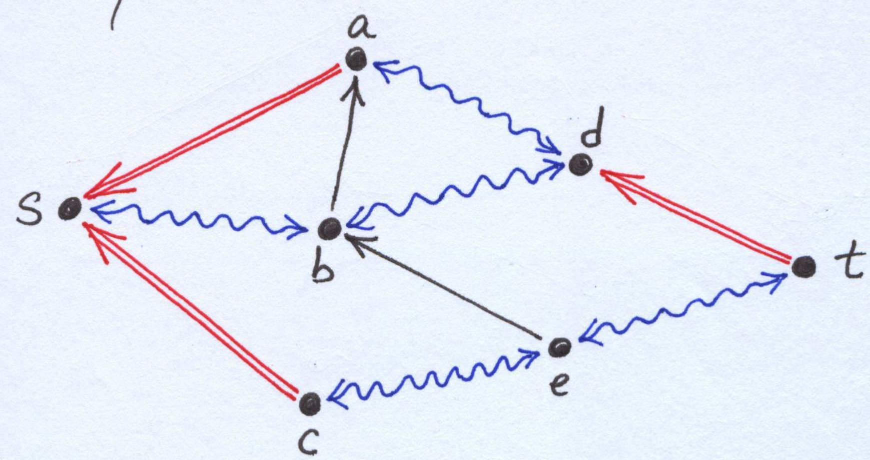
18 June. Augmenting paths for max flow — Example — 5

New flow:



Value of flow is 11.

Auxiliary network:



The auxiliary network does not contain a directed path from s to t, so there is no augmenting path, which means that our current flow is <u>optimal</u>.