15 June

The primal-dual method applied to the shortest-path problem [P&S §5.4]

Recall the node-arc LP formulation of the shortest-path problem (11 June):

vector of arc weights

flows along arcs

Primal: $\min \; c^T f$

$$\text{s.t.} \quad A f = \begin{array}{cl} \begin{bmatrix} +1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \leftarrow \text{row } s \text{ (source node)} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{all other rows (except } t\text{)} \end{array} \end{array} \quad n-1 \text{ rows in all.}$$

node-arc incidence matrix (without row t)

$$f \geq 0.$$

Note that we have omitted the constraint for node t (the terminal node), because it is redundant.

Dual: $\max \; \pi_s$

$$\text{s.t.} \quad \pi_i - \pi_j \leq c_{ij} \quad \text{for all arcs } (i,j) \in E$$

[i.e., $i \to j$]

$$\pi_i \text{ unrestricted for } i \neq t$$

[Because we deleted the constraint for t in the primal, so π_t doesn't really exist in the dual!] $\to \pi_t = 0.$

[Note: If all arc weights c_{ij} are nonnegative, $\pi = 0$ is a feasible dual solution, so <u>step 1</u> of primal-dual is easy.]

Recall that the set of <u>admissible</u> columns in the primal-dual algorithm is defined by

$$J = \{ j : j\text{th dual constraint is tight} \}.$$

So, in the context of the shortest-path problem, where dual constraints correspond to arcs in the graph, we get a set of <u>admissible arcs</u>:

$$J = \{ (i,j) \in E : \pi_i - \pi_j = c_{ij} \}.$$

<u>In step 2</u> of the primal-dual algorithm, we are searching for a feasible primal solution f such that $f_j = 0$ for all $j \notin J$. We do this by forming the restricted primal:

<u>Restricted primal:</u> $\quad \min \xi = \sum_{i=1}^{n-1} r_i$

$$\text{s.t.} \quad Af + r = \begin{array}{c} \begin{bmatrix} +1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{row } s \\ \\ \\ \\ \end{array}$$

flow along admissible arcs $\rightarrow$ $\quad f_j \geq 0 \quad$ for $j \in J$

flow along non-admissible arcs $\rightarrow$ $\quad f_j = 0 \quad$ for $j \notin J$
must be zero

$$r_i \geq 0 \quad \text{for } 1 \leq i \leq n-1.$$

15 June

Primal-dual for shortest path — 2

The dual of the restricted primal (DRP) is then:

DRP: $\max\ \omega = \pi_s$

s.t. $\pi_i - \pi_j \leq 0$ for arcs $(i,j) \in J$

$\pi_i \leq 1$ for all i

π_i unrestricted for $i \neq t$

$\pi_t = 0$.

This LP (the DRP) is easy to solve.

[Step 3]
- If there is a path from s to t using only arcs in J, then the constraint $\pi_t = 0$ and the constraints $\pi_i - \pi_j \leq 0$ for the arcs (i,j) along this path force $\pi_s \leq 0$, so an optimal solution is $\bar{\pi}_i = 0$ for all i, which yields the objective value 0. But this means that the optimal objective value for the restricted primal is also 0, so we succeeded in step 2 of the primal-dual algorithm ($\xi_{opt} = 0$), which means that our current dual solution π (and hence also the corresponding primal solution f from the restricted primal) is optimal. — i.e., any path from s to t using only arcs in J.

- Otherwise, there does not exist a path from s to t using only arcs in J. We can construct an optimal solution $\bar{\pi}$ to DRP as follows. Define

  $$W = \{\text{nodes } i : \text{there exists a path from } i \text{ to } t \text{ using only arcs in } J\}$$

  and let

  $$\bar{\pi}_i = \begin{cases} 0, & \text{if } i \in W; \\ 1, & \text{if } i \notin W. \end{cases}$$

  Note that $t \in W$ and $s \notin W$, so $\bar{\pi}_t = 0$ and $\bar{\pi}_s = 1$. This solution $\bar{\pi}$ is feasible because $\bar{\pi}_i - \bar{\pi}_j \leq 0$ for all arcs $(i,j) \in J$ [since $(i,j) \in J$, it is impossible to have $\bar{\pi}_i = 0$ and $\bar{\pi}_j = 1$].
  And $\bar{\pi}$ achieves an objective value of 1 in the DRP, which is clearly optimal because $\pi_s \leq 1$ is one of the constraints.
  — Note that $\bar{\pi}$ is not the <u>unique</u> optimal solution.

  So $\xi_{opt} = 1 > 0$, which means we continue to step 4.

<u>Step 4</u>: In the description of the general primal–dual algorithm, we saw that we should take

This is a multiplier. It has nothing to do with the node t. Collision of notation. Sorry. P&S use θ_1 for this.

$$t = \min_{\substack{j \notin J \text{ such that} \\ (A_j)^T \bar{\pi} > 0}} \left\{ \frac{c_j - (A_j)^T \pi}{(A_j)^T \bar{\pi}} \right\}. \qquad (\star)$$

Here $(A_j)^T \pi$ is the left-hand side of the jth constraint in the DRP.

Primal-dual for shortest path – 3

15 June

For the shortest-path problem, elements of J are arcs in the graph, and the left-hand side of the constraint in the DRP corresponding to an arc (i,j) is $\pi_i - \pi_j$. So $(\star)$ becomes

$$t = \min_{\substack{(i,j) \notin J \text{ such that} \\ \bar{\pi}_i - \bar{\pi}_j > 0}} \left\{ \frac{c_{ij} - (\pi_i - \pi_j)}{\bar{\pi}_i - \bar{\pi}_j} \right\}. \qquad (\star\star)$$

It is useful to have a name for this set of arcs over which we are minimizing (even though P&S give it no name), so define

$$K = \{ \text{arcs } (i,j) \notin J : \bar{\pi}_i - \bar{\pi}_j > 0 \}$$

and call these the <u>candidate arcs.</u>

Now, if $\bar{\pi}_i - \bar{\pi}_j > 0$, then by our particular construction of the solution $\bar{\pi}$ we must have $\bar{\pi}_i = 1$ and $\bar{\pi}_j = 0$. So really we can say

$$K = \{ \text{arcs } (i,j) \notin J : \bar{\pi}_i = 1 \text{ and } \bar{\pi}_j = 0 \},$$

i.e., K is the set of non-admissible arcs that point from a node i having $\bar{\pi}_i = 1$ to a node j having $\bar{\pi}_j = 0$: it's the set of non-admissible arcs from a node <u>outside</u> W to a node <u>inside</u> W.

This also means that $\bar{\pi}_i - \bar{\pi}_j = 1$ for all arcs $(i,j) \in K$, so the denominator in (★★) is 1, so

$$\boxed{t = \min_{(i,j)\in K} \left\{ c_{ij} - (\pi_i - \pi_j) \right\}.}$$

Then our new feasible dual solution π^* is $\pi^* = \pi + t\bar{\pi}$. Since $\bar{\pi}_i = 0$ for all $i \in W$ and $\bar{\pi}_i = 1$ for all $i \notin W$, this really just means:

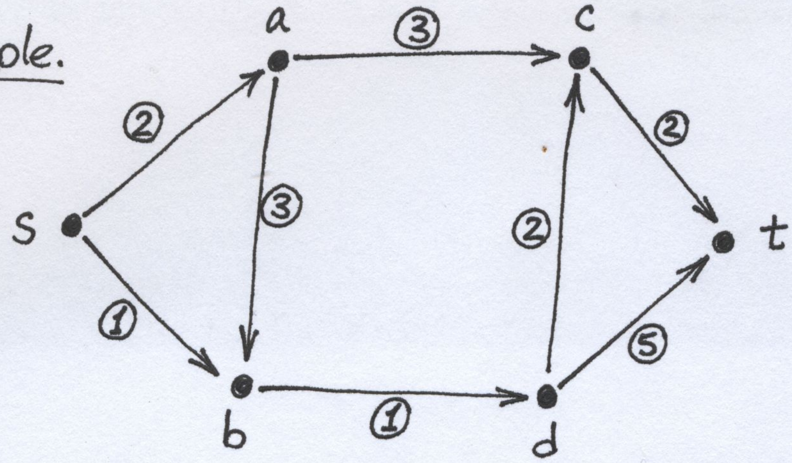
- If $i \in W$ (i.e., if t is reachable from i using only arcs in J), then $\pi_i^* = \pi_i$: the dual value of i does not change.

- If $i \notin W$, then $\pi_i^* = \pi_i + t$: the dual value of i increases by t.

Now we update J and go back to step 2.

15 June.

Example.

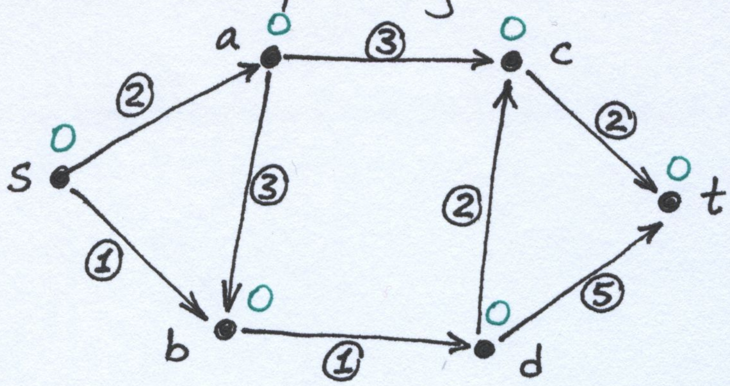


(Arc weights are circled.)

(Steps of the primal-dual algorithm.)

Step 1: Start with a feasible solution π to the dual.

Since all arc weights are nonnegative, $\pi = 0$ is a feasible dual solution. Let's write dual values in green above the corresponding nodes:



Step 2. We determine the admissible arcs:

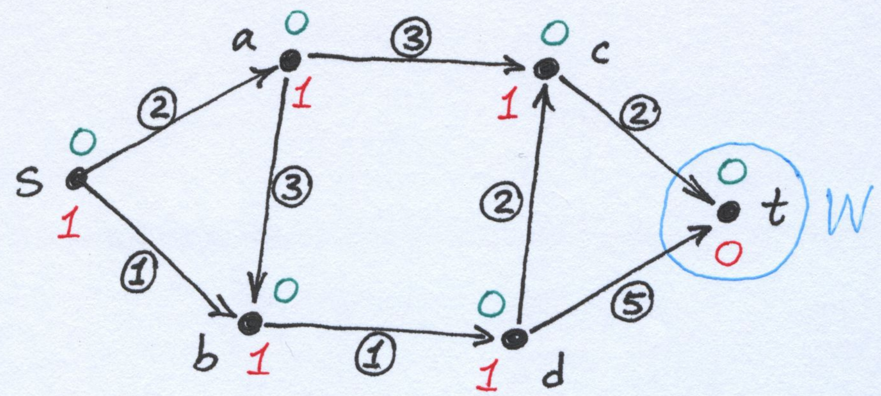
$$J = \{ (i,j) \in E : \pi_i - \pi_j = c_{ij} \} = \emptyset.$$

Note that admissible arcs are determined using the current dual solution π.

At the moment, the set of admissible arcs is empty.

15 June. Primal-dual for shortest path. Example — 2

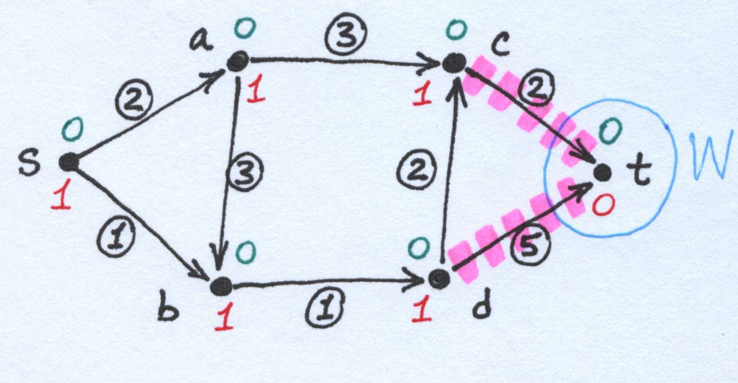
Clearly $\boxed{J}$ does not contain a path from s to t, so step 3 of the primal-dual algorithm does not apply ($\xi_{opt} > 0$). The set W of nodes from which t is reachable using arcs in $\boxed{J}$ contains only t itself, so our optimal solution $\overline{\pi}$ to the DRP is $\overline{\pi}_t = 0$, $\overline{\pi}_s = \overline{\pi}_a = \overline{\pi}_b = \overline{\pi}_c = \overline{\pi}_d = 1$. Let's write these values in red below the corresponding nodes:



Step 4. We determine the set of candidate arcs:

$$\boxed{K} = \{ \text{arcs } (i,j) \notin \boxed{J} : \overline{\pi}_i = 1 \text{ and } \overline{\pi}_j = 0 \} = \{ (c,t), (d,t) \}.$$

Note that the candidate arcs are determined using the optimal solution $\overline{\pi}$ to the DRP. Equivalently, they are the non-admissible arcs that point from a node outside W to a node inside W.

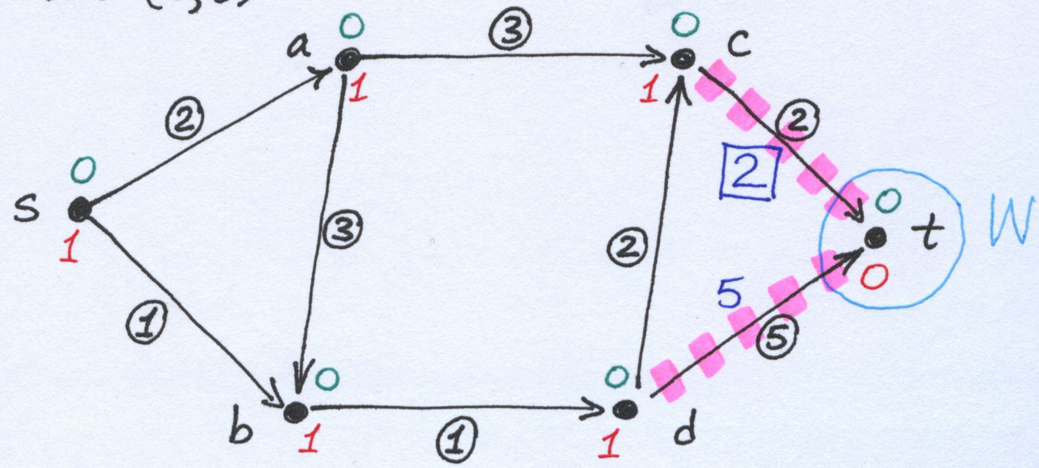


15 June. Primal-dual for shortest path. Example — 3

Now we compute $c_{ij} - (\pi_i - \pi_j)$ for all of the candidate arcs in K:

- For (c,t): $c_{ct} - (\pi_c - \pi_t) = 2 - (0 - 0) = 2$.
- For (d,t): $c_{dt} - (\pi_d - \pi_t) = 5 - (0 - 0) = 5$.

The minimum of these values is 2, achieved by the arc (c,t):

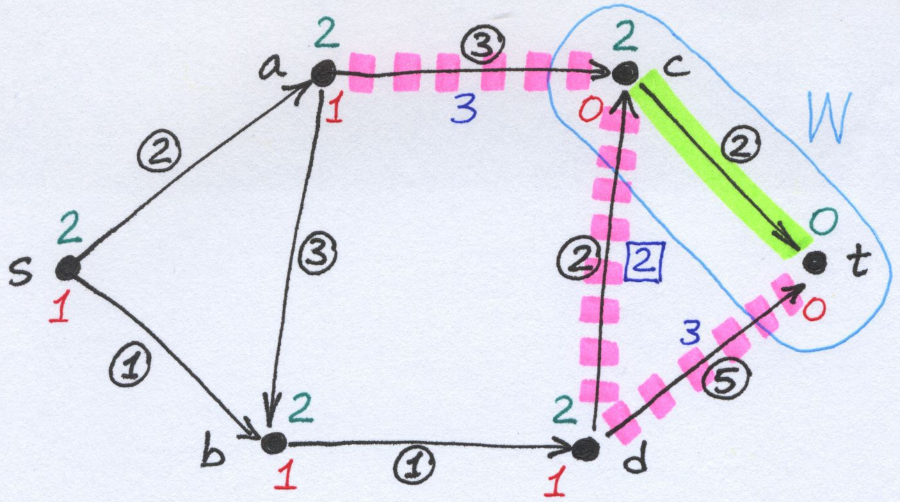


So $t = 2$. This means that our next dual solution π^* will be formed by adding 2 to the current dual values of the nodes not in W (so $\pi_s^* = \pi_a^* = \pi_b^* = \pi_c^* = \pi_d^* = 2$ and $\pi_t^* = 0$).

We go back to step 2 of the primal-dual algorithm and do it again.

15 June. Primal-dual for shortest path. Example — ④.

The result of the second iteration is:



<u>Key</u> (items in order of computation)

Dual solution π (written above nodes)

Admissible arcs J

Set W of nodes from which t is reachable using arcs in J

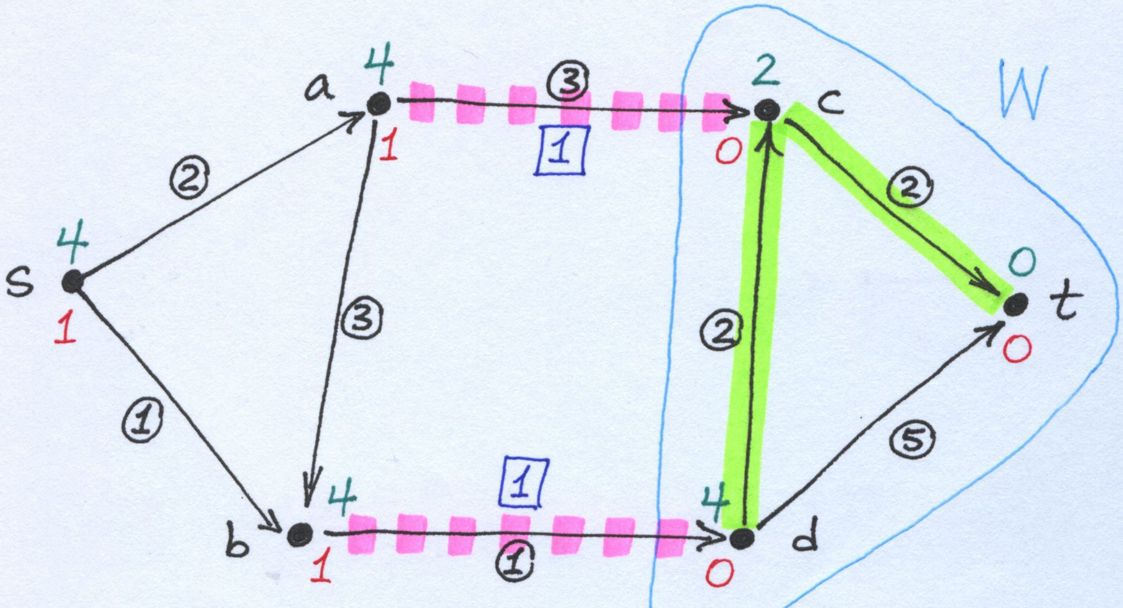
Optimal solution $\bar{\pi}$ to DRP (written below nodes)

Candidate arcs K

$c_{ij} - (\pi_i - \pi_j)$ for candidate arcs (minimum boxed)

So $t = 2$.

Adjust the dual solution accordingly, and perform a third iteration:

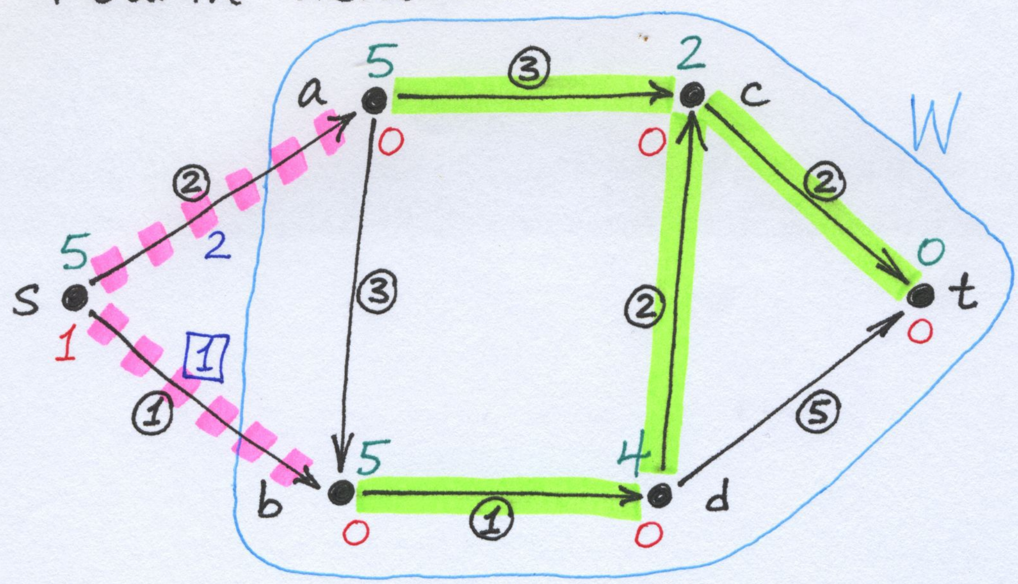


So $t = 1$.

Adjust dual solution. Continue.

15 June. Primal-dual for shortest path. Example — 5.

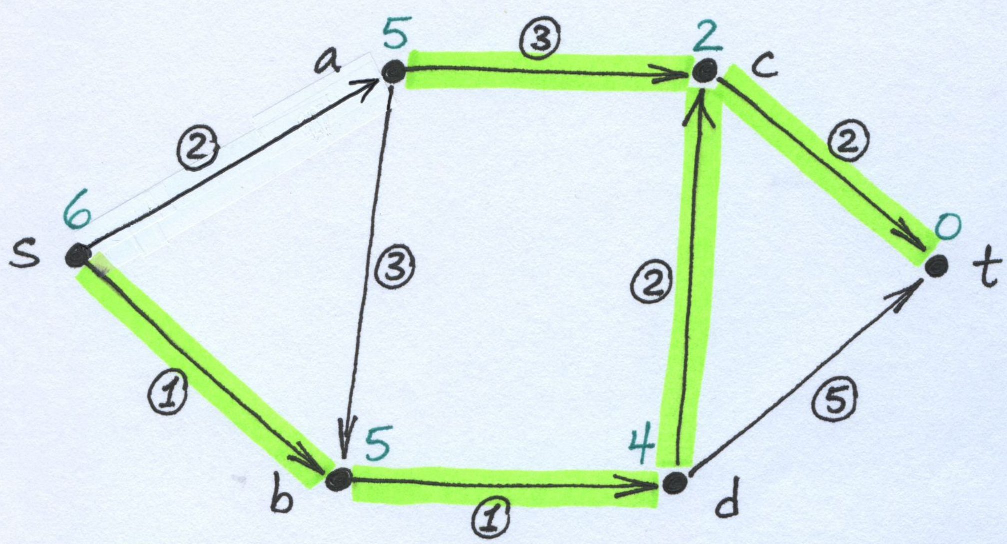
Fourth iteration:



(Note that we gained <u>two</u> admissible arcs in this iteration, because we had a tie for the minimum $c_{ij}-(\pi_i-\pi_j)$ in the previous iteration.)

So $t=1$. Adjust dual solution.

Fifth iteration:



Now J contains a path from s to t, so this path (s – b – d – c – t) is <u>optimal</u>.

The shortest distance from s to t is $\pi_s = 6$.

15 June

Observations.

- Once a node i enters W, the value of π_i remains fixed for the rest of the algorithm, because $\bar{\pi}_i$ will always be zero.

- Once an arc (i,j) becomes admissible (i.e., enters J), it remains admissible for the rest of the algorithm, because once we have $\pi_i - \pi_j = c_{ij}$ the values of π_i and π_j will always change by the same amount on each iteration (since $\bar{\pi}_i = \bar{\pi}_j$ for every arc $(i,j) \in J$ by our construction of $\bar{\pi}$).

- For $i \in W$, the value of π_i is the length of the shortest path from i to t. (Exercise: Prove this.)

- In each iteration, the algorithm proceeds by identifying the nodes not in W that are closest to W, and adding these nodes to W.

- The set W grows by at least one node in each iteration, so the algorithm cannot continue for more iterations than there are nodes in the graph.