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## Primal–dual algorithm [P&S Chap. 5]

Goal: Solve a min LP in standard form:

$$\begin{aligned} \min \quad & z = c^T x \\ \text{s.t.} \quad & Ax = b \geq 0 \\ & x \geq 0 \end{aligned} \qquad \text{[Primal LP]}$$

Note that we are assuming $b \geq 0$ here. If necessary, multiply some of the equations by $-1$ in order to make this true.

Dual LP:
$$\begin{aligned} \max \quad & w = b^T \pi \\ \text{s.t.} \quad & A^T \pi \leq c \\ & \pi \text{ unrestricted.} \end{aligned}$$

Recall complementary slackness: If $x$ and $\pi$ are feasible solutions to the primal and dual, respectively, then they are optimal if and only if

- for all $i$, either $\pi_i = 0$ or $\underbrace{a_i x = b_i}_{i\text{th primal constraint is tight}}$ (or both);
- for all $j$, either $x_j = 0$ or $\underbrace{(A_j)^T \pi = c_j}_{j\text{th dual constraint is tight.}}$ (or both).

Note that because the constraints in the primal are equalities, all primal constraints will always be tight for any feasible solution $x$, so the first complementary slackness condition will be satisfied automatically with no conditions on $\pi$.

Therefore, if we have a feasible solution $\pi$ to the dual, and we can find a feasible solution $x$ to the primal such that

$$x_j = 0 \quad \text{whenever} \quad (A_j)^T \pi < c_j,$$

then all the complementary slackness conditions will be satisfied, and we can conclude that both $\pi$ and $x$ are optimal.

This is the idea of the primal-dual algorithm:

1. Start with a feasible dual solution $\pi$.
2. Search for a feasible primal solution $x$ such that $x_j = 0$ whenever $(A_j)^T \pi < c_j$.
3. If we succeed in step 2, we're done.
4. Otherwise, use the "best" solution found in step 2 (i.e., "closest to being feasible") to adjust (improve) the dual solution, and repeat.

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# Primal-dual algorithm — 2

**Step 1:** Start with a feasible dual solution $\pi$.

- For certain problems, it is easy to come up with an initial feasible dual solution.
  - E.g., if $c \geq 0$, then $\pi = 0$ is feasible.
- Otherwise, use a trick described in P&S §5.2, attributed to Beale; see P&S for details.

**Step 2:** Search for a feasible primal solution $x$ such that $x_j = 0$ whenever $(A_j)^T \pi < c_j$.

Define $J = \{ j : (A_j)^T \pi = c_j \}$.

- This is the set of indices of dual constraints that are tight, so the corresponding variables in the primal can have any nonnegative values.
- The other primal variables, $x_j$ for $j \notin J$, must be fixed to zero.
- So $J$ is called the set of admissible columns: these are the columns (i.e., variables) in the primal tableau that are allowed to become basic.

So what we are searching for in step 2 is an x that satisfies

$$\sum_{j \in J} a_{ij} x_j = b_i \quad \text{for all } 1 \leq i \leq m$$

$$x_j \geq 0 \quad \text{for } j \in J$$

$$x_j = 0 \quad \text{for } j \notin J.$$

How do we search for such an x? We add artificial variables $r_i$ for $1 \leq i \leq m$ on the left-hand sides of the equations above, and then minimize the sum of the $r_i$'s. (This is just like Phase I of the two-phase simplex algorithm.)

So we have the following LP, which is called the <u>restricted primal</u>:

$$\min \xi = \sum_{i=1}^{m} r_i$$

$$\text{s.t.} \quad \Big(\sum_{j \in J} a_{ij} x_j\Big) + r_i = b_i \quad \text{for } 1 \leq i \leq m$$

$$\left.\begin{array}{l} x_j \geq 0 \quad \text{for } j \in J \\ x_j = 0 \quad \text{for } j \notin J \end{array}\right\} \text{View this as including only the variables from the primal corresponding to J.}$$

$$r_i \geq 0 \quad \text{for all } i$$

[Note: Both the $x_j$'s and the $r_i$'s are variables here.]

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Primal–dual algorithm – 3

— Note that because we assumed $b \geq 0$ at the beginning, the solution $x=0$, $r=b$ is always feasible, so we don't have to worry about infeasibility or finding an initial bfs.

— The restricted primal can be solved using the minimizing simplex algorithm described in P&S, or by solving its dual (coming up shortly) using the maximizing simplex algorithm.

— But the real strength of the primal–dual algorithm is that for many problems the restricted primal is significantly easier to solve than the primal, and it is possible to solve the restricted primal more easily or more efficiently than by going through the simplex algorithm.

— Also, the successive restricted primals that are encountered on each iteration of the primal–dual algorithm are very similar, so the solution to one can be used as a starting point for the solution to the next — see P&S §4.1, 5.3.

Step 3. If we succeed in step 2, we're done.

— We succeed in step 2 if the optimal objective value of the restricted primal is $\xi_{opt} = 0$, because then all the $r_i$'s are zero and we have an $x$ that satisfies the desired conditions.

Step 4. Otherwise, use the "best" solution found in step 2 (i.e., "closest to being feasible") to adjust (improve) the dual solution, and repeat.

Consider the dual of the restricted primal (DRP):

$$\max \ \omega = b^T \pi$$

$$\text{s.t.} \quad (A_j)^T \pi \leq 0 \quad \text{for } j \in J$$

only the constraints from the dual corresponding to $J$ (with RHS zero)

$$\pi_i \leq 1 \quad \text{for } 1 \leq i \leq m$$

from the artificial variables $r_i$

$$\pi_i \ \text{unrestricted}.$$

If we are in step 4, then $\xi_{opt} > 0$ (because if $\xi_{opt} = 0$ we would be done, in step 3).

Let $\bar{\pi}$ be the optimal solution to the DRP.

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# Primal-dual algorithm — 4

We will adjust our dual solution by adding a multiple of $\bar{\pi}$:

$$\underbrace{\pi^*}_{\text{new dual solution}} = \underbrace{\pi}_{\text{old dual solution}} + \underbrace{t}_{\text{scalar}} \underbrace{\bar{\pi}}_{\text{optimal solution to DRP}}$$

The objective value of this new dual solution will be

$$b^T\pi^* = b^T\pi + t\, b^T\bar{\pi}$$

$$= b^T\pi + t\, \xi_{opt}.$$

[Because $\bar{\pi}$ is an optimal solution to DRP, so its objective value $b^T\bar{\pi}$ is equal to the optimal objective value of the restricted primal, $\xi_{opt}$.]

Since $\xi_{opt} > 0$, we should choose $t > 0$ so that the objective value of the new dual solution is larger (better) than that of the old dual solution.

Now consider the dual constraints. We need

(★) $$(A_j)^T\pi^* = (A_j)^T\pi + t(A_j)^T\bar{\pi} \le c_j \quad \text{for } 1 \le j \le n.$$

We know $(A_j)^T\pi \le c_j$ for all $j$, because $\pi$ is feasible.

If $(A_j)^T\bar{\pi} \le 0$ for some $j$, then (★) will be satisfied for that $j$ for any $t > 0$.

We know that $(A_j)^T \bar{\pi} \le 0$ for all $j \in J$ because those are constraints in the DRP. So we don't need to worry about $j \in J$. We just need to concern ourselves with satisfying ($\star$) for $j \notin J$.

If $(A_j)^T \bar{\pi} \le 0$ for <u>all</u> $j \notin J$, then ($\star$) will be satisfied for all $1 \le j \le n$ for any $t > 0$. In other words, $t$ can be made arbitrarily large. Since the objective value of $\pi^*$ is $b^T\pi + t\,\xi_{opt}$ and $\xi_{opt} > 0$, this means that the objective value of $\pi^*$ can be made arbitrarily large. Hence the dual is unbounded, which means that the primal is <u>infeasible.</u>

<u>Theorem.</u> If $\xi_{opt} > 0$ and $(A_j)^T \bar{\pi} \le 0$ for all $j \notin J$, then the primal LP is infeasible.

Otherwise, we have $(A_j)^T \bar{\pi} > 0$ for some $j \notin J$. We need

$$\left.\begin{aligned} &(A_j)^T \pi + t (A_j)^T \bar{\pi} \le c_j, \\ \text{i.e.,}\quad & t \le \frac{c_j - (A_j)^T \pi}{(A_j)^T \bar{\pi}} \end{aligned}\right\} \text{ for all } j \notin J \text{ such that } (A_j)^T \bar{\pi} > 0.$$

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# Primal-dual algorithm — 5

So take

$$t = \min_{\substack{j \notin J \text{ such that} \\ (A_j)^T \bar{\pi} > 0}} \left\{ \frac{c_j - (A_j)^T \pi}{(A_j)^T \bar{\pi}} \right\}.$$

For all $j \notin J$, we know $c_j - (A_j)^T \pi > 0$, because $(A_j)^T \pi \leq c_j$ is a constraint in the dual, and if $(A_j)^T \pi = c_j$ then $j$ would be in $J$ (by the definition of $J$). So the numerators of all these fractions are strictly positive. And the denominators are positive, so $t > 0$.

So our new dual solution is

$$\pi^* = \pi + t\,\bar{\pi},$$

where $t$ is given above. The new dual objective value is

$$b^T \pi^* = b^T \pi + t\, b^T \bar{\pi} = b^T \pi + t\, \xi_{opt} > b^T \pi,$$

so the new dual solution is strictly better than the old one.

Now go back to step 2, using $\pi^*$ as the dual solution.