Overview of the transportation algorithm

10 June

1. Form an initial basic feasible solution $\{x_{ij}\}$ to the primal LP.

2. Use complementary slackness to find a corresponding solution $\{v_i, w_j\}$ to the dual LP.

   (By complementary slackness, when x_{ij} is basic, we should have $v_i + w_j = c_{ij}$.)

3. Compute test values $c_{ij} - v_i - w_j$.

   (Test values for i,j corresponding to basic x_{ij} will always be zero, because of step 2. So we just need test values for nonbasic x_{ij}.)

4. If all test values are nonnegative, <u>STOP</u>: current solution $\{x_{ij}\}$ is optimal.

   (By second theorem.)

5. Otherwise, choose a negative test value (as a rule of thumb, the <u>most</u> negative one) and increase the corresponding x_{ij} as much as possible to decrease the cost. This is called a <u>pivot</u>. (Example shortly.) Then go back to step 2.

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The transportation tableau

Rows correspond to origins.
Columns correspond to destinations.

Example (same as previously):

Supply

A	10
B	15
C	10

Demand

W	8
X	12
Y	7
Z	8

Per-unit costs

FROM \ TO	W	X	Y	Z
A	$12	9	11	7
B	15	8	10	9
C	7	4	6	11

Blank transportation tableau with this information:

	W:8	X:12	Y:7	Z:8
A:10	12	9	11	7
B:15	15	8	10	9
C:10	7	4	6	11

Squares in this grid correspond to variables in the LP, so we will have basic and nonbasic squares corresponding to basic and nonbasic variables.

First step: Get an initial basic feasible solution.

Method:

- Satisfy each demand one by one, from left to right.
- For each demand, take units from each supply, top to bottom, until the demand is met; but once a supply has been exhausted, it cannot be used to satisfy later demands.
- If necessary, add one or more basic squares with a value of zero. (More about this later.)

For this example, our initial basic feasible solution is:

	W: 8	X: 12	Y: 7	Z: 8
A: 10	8 $\boxed{12}$	2 $\boxed{9}$	$\boxed{11}$	$\boxed{7}$
B: 15	$\boxed{15}$	10 $\boxed{8}$	5 $\boxed{10}$	$\boxed{9}$
C: 10	$\boxed{7}$	$\boxed{4}$	2 $\boxed{6}$	8 $\boxed{11}$

Objective value (total cost):

$8(12) + 2(9) + 10(8)$
$+ 5(10) + 2(6)$
$+ 8(11) = \underline{\underline{\$344}}.$

Basic squares are shaded. Number in lower left corner is value of corresponding basic variable X_{ij}, i.e., number of units to be sent from origin i to destination j.

Transportation tableau — 2

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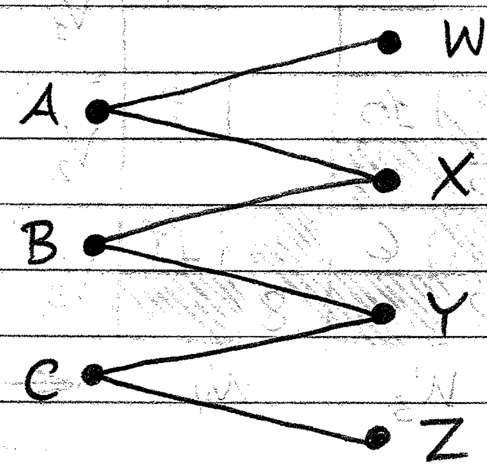
<u>Note:</u> Suppose we make a bipartite graph with vertices representing origins on the left and vertices representing destinations on the right, with an edge joining origin i to destination j if and only if the variable x_{ij} is basic:



Observe that this graph is a <u>tree</u>: it is connected (there is a path between any two vertices) and it has no cycles. (This particular graph happens to be a path graph, but that will not be true in general.)

This is an important fact. Basic solutions for the transportation problem will always correspond to trees.

★ If necessary, include one or more basic squares with value zero in order to get a tree.

Now that we have an initial basic feasible solution, we use complementary slackness to find a corresponding solution $\{v_i, w_j\}$ to the dual LP. Complementary slackness implies that when x_{ij} is basic, we should have $v_i + w_j = c_{ij}$.

v_i's correspond to origins

	W	X	Y	Z	
A	12 (8)	9 (2)	11	7	v_1
B	15	8 (10)	10 (5)	9	v_2
C	7	4	6 (2)	11 (8)	v_3
	w_1	w_2	w_3	w_4	

w_j's correspond to destinations

From complementary slackness:

$$
\begin{array}{ll}
(x_{AW}) & v_1 + w_1 = 12 \\
(x_{AX}) & v_1 + w_2 = 9 \\
(x_{BX}) & v_2 + w_2 = 8 \\
(x_{BY}) & v_2 + w_3 = 10 \\
(x_{CY}) & v_3 + w_3 = 6 \\
(x_{CZ}) & v_3 + w_4 = 11
\end{array}
$$

We have six equations in seven unknowns, so we have one free variable — we can set any one of these variables to any value we like, and then solve for the remaining six.

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Transportation tableau — 3

For simplicity, let's set $v_1 = 0$. Solving for the remaining six dual variables, we get

	W	X	Y	Z	
A	$\boxed{12}$ 8	$\boxed{9}$ 2	$\boxed{11}$	$\boxed{7}$	$v_1 = 0$
B	$\boxed{15}$	$\boxed{8}$ 10	$\boxed{10}$ 5	$\boxed{9}$	$v_2 = -1$
C	$\boxed{7}$	$\boxed{4}$	$\boxed{6}$ 2	$\boxed{11}$ 8	$v_3 = -5$
	$w_1 = 12$	$w_2 = 9$	$w_3 = 11$	$w_4 = 16$	

Now, for the <u>nonbasic</u> squares, we calculate test values, $c_{ij} - v_i - w_j$, and write them in the lower right corner:

	W	X	Y	Z	
A	$\boxed{12}$ 8	$\boxed{9}$ 2	$\boxed{11}$ 0	$\boxed{7}$ −9	$v_1 = 0$
B	$\boxed{15}$ 4	$\boxed{8}$ 10	$\boxed{10}$ 5	$\boxed{9}$ −6	$v_2 = -1$
C	$\boxed{7}$ 0	$\boxed{4}$ 0	$\boxed{6}$ 2	$\boxed{11}$ 8	$v_3 = -5$
	$w_1 = 12$	$w_2 = 9$	$w_3 = 11$	$w_4 = 16$	

We have negative test values, so this solution is not <u>optimal</u>. Choose the most negative (−9 in AZ) and <u>pivot</u> there.

Pivoting. The test value -9 in the AZ square indicates that if we make x_{AZ} positive (by bringing it into the basis) then we will decrease our total cost. So we want to increase the value of x_{AZ} from its current value of zero to some new value t.

But unless we adjust the values of the other x_{ij}'s, we will mess up our row and column sums. The total of the x_{ij}'s in the Z column must be 8 (to satisfy Z's demand), so if we increase x_{AZ} to t then we need to decrease x_{CZ} to the value $8-t$.

Now that adjustment messed up the total for the C row, so we need to increase the value of x_{CY} to $2+t$.

Following this chain of reasoning:

- (Y column) Decrease x_{BY} to $5-t$.
- (B row) Increase x_{BX} to $10+t$.
- (X column) Decrease x_{AX} to $2-t$.

This last adjustment also fixed the A row sum that we messed up by increasing x_{AZ} at the beginning. So now we're all good!

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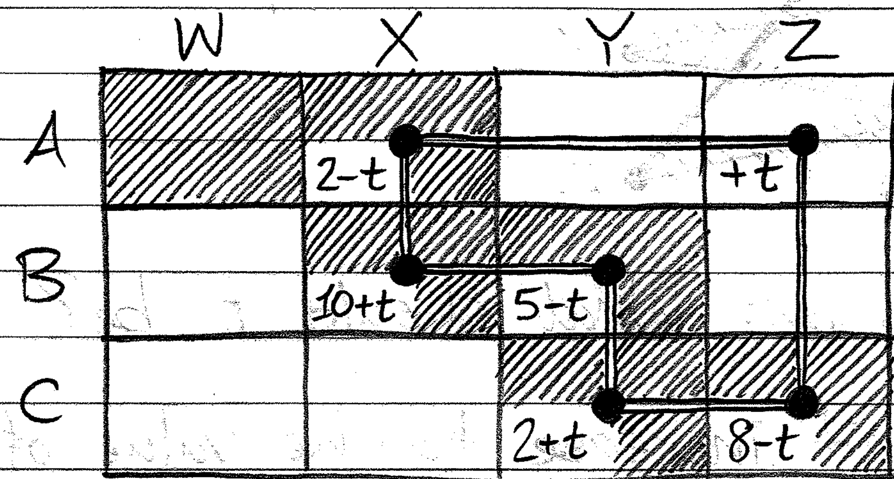
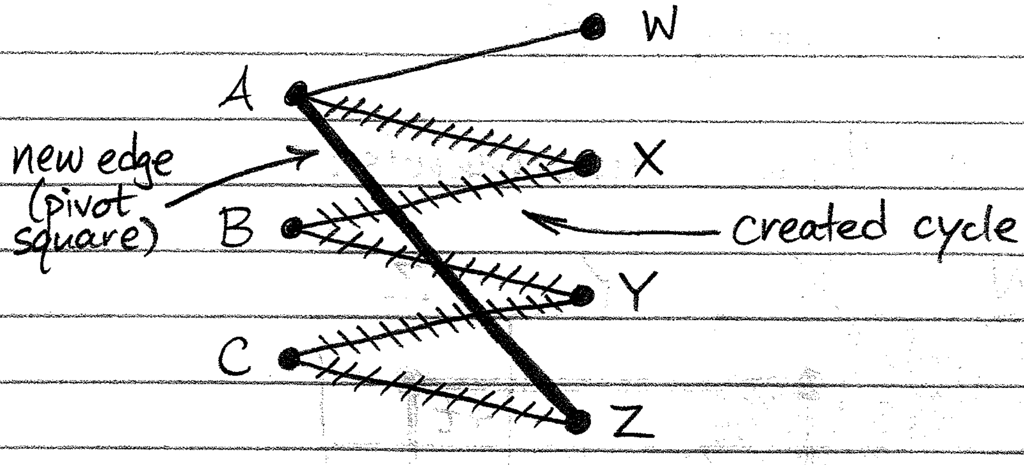
Transportation pivoting — 2

Summary of these adjustments:



Note that these adjustments follow a circuit of "alternating-direction rook's moves" (alternating vertical and horizontal motions) that, except for the pivot square AZ, turns 90° only on <u>basic</u> squares. The adjustments alternate +t, −t, +t, −t, … at the "corners" of this circuit, starting with +t at the pivot square.

There will always be <u>exactly one</u> such circuit in the tableau for any given pivot square. This is <u>because</u> the basic squares correspond to the edges of a <u>tree</u>, and adding one edge (i.e., the pivot square) to a tree creates a unique cycle:



Next question: What should t be?

Every unit increase to the value of x_{AZ} will decrease our total cost by \$9 (that's what the test value means), so we want to make t as large as possible.

But if t is too large, the values of x_{AX}, x_{BY}, and x_{CZ}, which are $2-t$, $5-t$, and $8-t$, respectively, will become negative, which violates their domains. The first to become negative will be $x_{AX}=2-t$, so the greatest we can make t is 2.

(Note that $x_{BX}=10+t$ and $x_{CY}=2+t$ do not place restrictions on t, because they will not become negative.)

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Transportation pivoting — 3

So, taking $t=2$, our new basic feasible solution is

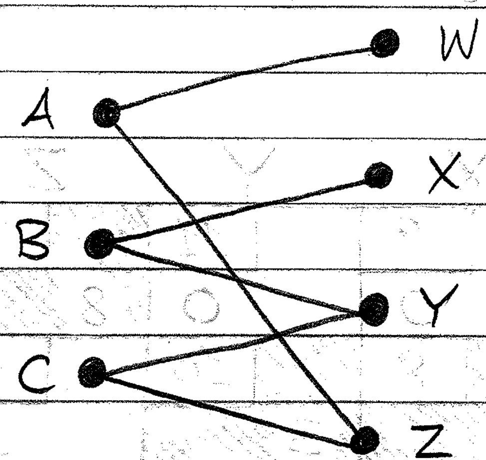
AX: fell out of basis (x_{AX} became zero)

x_{AW} is still 8 because AW was not part of the pivot circuit.

AZ: new basic square

	W	X	Y	Z
A	cost 12; 8	cost 9	cost 11	cost 7; 2
B	cost 15	cost 8; 12	cost 10; 3	cost 9
C	cost 7	cost 4	cost 6; 4	cost 11; 6

Objective value: $8(12)+2(7)+12(8)+3(10)+4(6)+6(11) = \underline{\underline{\$326}}$ (improved)



Still a <u>tree</u>. (connected, acyclic)

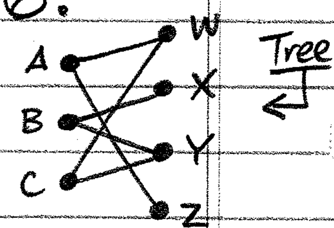
Compute dual variables and test values:

	W	X	Y	Z	
A	cost 12; 8	cost 9; 9	cost 11; 9	cost 7; 2	$V_1=0$ (arbitrarily)
B	cost 15; −5	cost 8; 12	cost 10; 3	cost 9; −6	$V_2=8$
C	cost 7; −9	cost 4; 0	cost 6; 4	cost 11; 6	$V_3=4$
	$W_1=12$	$W_2=0$	$W_3=2$	$W_4=7$	

Still have negative test values, so still not optimal. Pivot on CW. Pivot circuit goes CW – AW – AZ – CZ – CW:

	W	X	Y	Z
A	8-t			2+t
B				
C	t			6-t

So largest possible value of t is 6.
Next tableau:



	W	X	Y	Z	
A	12 / 2	9 / 0	11 / 0	7 / 8	$V_1 = 0$
B	15 / 4	8 / 12	10 / 3	9 / 3	$V_2 = -1$
C	7 / 6	4 / 0	6 / 4	11 / 9	$V_3 = -5$
	$W_1 = 12$	$W_2 = 9$	$W_3 = 11$	$W_4 = 7$	

All test values are nonnegative, so this solution is optimal:

2 units A→W, 8 units A→Z, 12 units B→X,
3 units B→Y, 6 units C→W, 4 units C→Y.

Optimal objective value (cost): $2(12)+8(7)+12(8)+3(10)+6(7)+4(6)$
$= \underline{\$272}$.