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Transportation problem

This presentation is from Walker, Introduction to Mathematical Programming.

- The problem is also called the Hitchcock problem: see P&S §7.4–7.5.
- Also discussed in Jungnickel and in Korte/Vygen (see syllabus).

Given:

- A set of m origins, the ith origin having a supply of a_i units. [e.g., warehouses]
- A set of n destinations, the jth destination having a demand of b_j units. [e.g., retail stores]
- A table of transportation costs: the cost c_{ij} is the cost to send one unit from origin i to destination j.

Objective: Determine the least expensive way to satisfy the demand at every destination using the supplies at the origins.

Assumption: $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, i.e., total supply equals total demand. If this is not true, add a fictitious demand to absorb the excess supply (or a fictitious supply to satisfy the excess demand), and set costs to or from this fictitious entity so that they make sense for the particular problem being solved.

Example. Three origins (A, B, C) and four destinations (W, X, Y, Z).

Supply	
A	10
B	15
C	10

Demand	
W	8
X	12
Y	7
Z	8

Costs

(per unit) FROM ↓ / TO →	W	X	Y	Z
A	\$12	9	11	7
B	15	8	10	9
C	7	4	6	11

LP formulation

Variables: One variable x_{ij} for each origin i and each destination j, indicating the number of units to be sent from i to j. (So there are mn variables in all.)

— Domain: $x_{ij} \geq 0$ for all i, j.

Objective: Minimize total transportation costs.

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Constraints:

- For each origin i, all supply is used: $\sum_{j=1}^{n} x_{ij} = a_i$.
- For each destination j, all demand is met: $\sum_{i=1}^{m} x_{ij} = b_j$.

Note: These are equality constraints because of our assumption that total supply equals total demand.

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$$\min\ 12x_{AW}+9x_{AX}+11x_{AY}+7x_{AZ}+15x_{BW}+8x_{BX}+10x_{BY}+9x_{BZ}+7x_{CW}+4x_{CX}+6x_{CY}+11x_{CZ}$$

$$
\begin{array}{llr}
 & & \text{Dual var} \\
\text{s.t.} & & \\
\text{supply constraints} & x_{AW}+x_{AX}+x_{AY}+x_{AZ} = 10 & (v_1) \\
 & x_{BW}+x_{BX}+x_{BY}+x_{BZ} = 15 & (v_2) \\
 & x_{CW}+x_{CX}+x_{CY}+x_{CZ} = 10 & (v_3) \\
\text{demand constraints} & x_{AW}+x_{BW}+x_{CW} = 8 & (w_1) \\
 & x_{AX}+x_{BX}+x_{CX} = 12 & (w_2) \\
 & x_{AY}+x_{BY}+x_{CY} = 7 & (w_3) \\
 & x_{AZ}+x_{BZ}+x_{CZ} = 8 & (w_4)
\end{array}
$$

All variables nonnegative.

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Dual LP for example transportation problem.

Dual variables:

- $v_1, \ldots, v_m$ corresponding to supply constraints.
- $w_1, \ldots, w_n$ Corresponding to demand constraints.

$$
\begin{array}{llllllllll}
\max & 10v_1 + 15v_2 + 10v_3 + 8w_1 + 12w_2 + 7w_3 + 8w_4 \\
\text{s.t.} & v_1 & & & +\, w_1 & & & & \le 12 \\
& v_1 & & & & +\, w_2 & & & \le 9 \\
& v_1 & & & & & +\, w_3 & & \le 11 \\
& v_1 & & & & & & +\, w_4 & \le 7 \\
& & v_2 & & +\, w_1 & & & & \le 15 \\
& & v_2 & & & +\, w_2 & & & \le 8 \\
& & v_2 & & & & +\, w_3 & & \le 10 \\
& & v_2 & & & & & +\, w_4 & \le 9 \\
& & & v_3 & +\, w_1 & & & & \le 7 \\
& & & v_3 & & +\, w_2 & & & \le 4 \\
& & & v_3 & & & +\, w_3 & & \le 6 \\
& & & v_3 & & & & +\, w_4 & \le 11
\end{array}
$$

All variables unrestricted.

In general, for an instance of the transportation problem having supplies $a_1, \ldots, a_m$, demands $b_1, \ldots, b_n$, and transportation costs c_{ij}, we have:

Primal LP:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \sum_{j=1}^{n} x_{ij} = a_i \quad \text{for all } i \quad [\text{supply}]$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{for all } j \quad [\text{demand}]$$

$$x_{ij} \geq 0 \quad \text{for all } i,j$$

Dual LP:

$$\max \sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{n} b_j w_j$$

$$\text{s.t. } v_i + w_j \leq c_{ij} \quad \text{for all } i,j$$

$$v_i, w_j \text{ unrestricted.}$$

Note: The constraints in the primal LP are redundant: any $m+n-1$ of them imply the remaining one. So the rank of the coefficient matrix is $m+n-1$. This means that a basic solution will have $\underline{\underline{m+n-1}}$ basic variables.

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Theorem. Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be any constants. Consider two instances of the transportation problem:

(i) supplies $a_1, \ldots, a_m$; demands $b_1, \ldots, b_n$; costs c_{ij}.

(ii) supplies $a_1, \ldots, a_m$; demands $b_1, \ldots, b_n$; costs $c_{ij} - v_i - w_j$.

In other words, the two instances are the same, except that the cost c_{ij} in the first instance becomes $c_{ij} - v_i - w_j$ in the second. Then:

(a) The objective functions of the primal LPs for these two instances differ by a constant, i.e., the difference between the two objective values for any solution $\{x_{ij}\}$ is a constant K that does not depend on $\{x_{ij}\}$.

(b) A solution $\{x_{ij}\}$ is optimal for one of these instances if and only if it is optimal for the other.

Proof.

(a) The objective function in (ii) is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - v_i - w_j) x_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) v_i - \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right) w_j$$

$$= \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}}_{\text{Objective function in (i).}} \underbrace{- \sum_{i=1}^{m} a_i v_i - \sum_{j=1}^{n} b_j w_j}_{\text{Constant because } a_i, v_i, b_j, w_j \text{ are constants.}}$$

(b) This is clear, because the primal LPs for the two instances have exactly the same constraints, so a solution $\{x_{ij}\}$ is feasible for one if and only if it is feasible for the other, and the objective functions differ by a constant. $\square$

<u>Theorem.</u> For an instance of the transportation problem having costs c_{ij}, and a feasible solution $\{x_{ij}\}$ to that instance, if there exist values $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ such that

- $c_{ij} - v_i - w_j = 0$ whenever $x_{ij} > 0$ and
- $c_{ij} - v_i - w_j \geq 0$ whenever $x_{ij} = 0$,

then $\{x_{ij}\}$ is optimal.

<u>Proof 1.</u> Consider the instance in which the costs are $c_{ij} - v_i - w_j$. All costs are nonnegative in this modified instance, and the objective value of $\{x_{ij}\}$ is zero, so $\{x_{ij}\}$ is clearly optimal for this instance. By part (b) of the previous theorem, $\{x_{ij}\}$ is optimal for the original instance too. $\square$

<u>Proof 2.</u> Because $v_i + w_j \leq c_{ij}$ for all i, j, the values of v_i and w_j form a feasible solution to the dual LP. The conditions in the theorem mean that the complementary slackness equations hold, so $\{x_{ij}\}$ is an optimal solution to the primal LP (and $\{v_i, w_j\}$ is an optimal solution to the dual). $\square$

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<u>Defn.</u> The values $c_{ij} - v_i - w_j$ are called <u>test values.</u>

— Test values play a role similar to entries in the objective row in the simplex tableau: By part (a) of the first theorem, we are equivalently minimizing

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - v_i - w_j)\, x_{ij},$$

so if $c_{ij} - v_i - w_j < 0$ for some i, j, then we can decrease cost by increasing x_{ij} (i.e., by making x_{ij} basic).

— In fact, the test values are not just <u>similar</u> to the entries in the objective row —— they <u>ARE</u> the entries in the objective row! (For the minimizing simplex algorithm, in which the objective row represents an equation where z has a coefficient of -1: the variant of the simplex algorithm described by P&S.)