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[P&S §2.1, page 28]

Nonstandard variable domains.

- Standard form of an LP requires all variables to be nonnegative.

- This is also an important assumption in the simplex algorithm.
  - Note that the variable domains don't make an explicit appearance in the algorithm, unlike the constraints. That's because the assumption that all vars are nonnegative is built into the algorithm itself.

Recall: Three possible variable domains in an LP:

$$\underbrace{x \geq 0,}_{\text{Standard.}} \quad x \leq 0, \quad x \text{ unrestricted.}$$

Technique:

- To handle a nonpositive variable $x \leq 0$, introduce a new variable $\bar{x} = -x$. Then $\bar{x} \geq 0$. Rewrite the entire LP in terms of $\bar{x}$ instead of x.

$\longrightarrow$

- To handle an <u>unrestricted</u> variable x, introduce two new variables x^+ and x^- with the domains $x^+ \geq 0$, $x^- \geq 0$. Make the substitution $x = x^+ - x^-$. Rewrite the whole LP in terms of x^+ and x^- instead of x.

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General constraints in the simplex algorithm.

Example. $\max\ 7x_1 + 5x_2$

$$\begin{aligned} \text{s.t. } 3x_1 + 2x_2 &\le 10 \\ x_1 + x_2 &\ge 4 \\ -x_1 + x_2 &= 3 \\ x_1 \ge 0,\ x_2 &\ge 0. \end{aligned}$$

One way to convert this to standard form is to negate the second constraint to get

$$-x_1 - x_2 \le -4$$

and then introduce slack variables for the first two constraints:

$$\begin{aligned} \max\ & 7x_1 + 5x_2 \\ \text{s.t. } & 3x_1 + 2x_2 + s_1 = 10 \\ & -x_1 - x_2 + s_2 = -4 \\ & -x_1 + x_2 = 3 \\ & x_1 \ge 0,\ x_2 \ge 0,\ s_1 \ge 0,\ s_2 \ge 0. \end{aligned}$$

Then the initial simplex tableau would be:

x_1	x_2	s_1	s_2	z	RHS
-7	-5	0	0	1	0
3	2	1	0	0	10
-1	-1	0	1	0	-4
-1	1	0	0	0	3

Problems with this.

- There is no "obvious basis." The s_1 and s_2 columns are two columns of the identity matrix, but we don't have another obvious column to complete the basis.

- Even if we did have an "obvious basis," there is a negative entry in the RHS column, which means one of our basic variables will have a negative value.

So we'll use a different technique to convert the LP to standard form.

slack variable
surplus variable
artificial variables

$$
\begin{aligned}
\max\ & 7x_1 + 5x_2 \\
\text{s.t. } & 3x_1 + 2x_2 + s_1 = 10 \\
& x_1 + x_2 - p_2 + a_2 = 4 \\
& -x_1 + x_2 + a_3 = 3 \\
& x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ p_2 \geq 0,\ a_2 \geq 0,\ a_3 \geq 0.
\end{aligned}
$$

- Note that the subscripts of slack, surplus, and artificial variables indicate <u>which constraint</u> they correspond to. This will be useful later.

- If the artificial variables are <u>zero</u>, then we get a feasible solution to the original LP.

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General constraints — 2

With this formulation, the initial simplex tableau is

x_1	x_2	s_1	p_2	a_2	a_3	z	RHS
−7	−5	0	0	0	0	1	0
3	2	1	0	0	0	0	10
1	1	0	−1	1	0	0	4
−1	1	0	0	0	1	0	3

<u>Observe:</u>

- We have an "obvious basis": $\{s_1, a_2, a_3\}$. So the initial basic solution is $x_1=0,\ x_2=0,\ s_1=10,\ p_2=0,\ a_2=4,\ a_3=4.$

- None of the entries in the RHS column is negative, so none of the variable domains is violated by this basic solution.

<u>Unfortunately</u>, $x_1=0, x_2=0$ is <u>not</u> a feasible solution to the original LP, because the last two constraints are violated (because the artificial variables have nonzero values).

- So we will need to fix this problem somehow.

Before we do that, though, let's review the technique so far:

- First, all constraints need to have nonnegative right-hand sides (to avoid negative entries in RHS column).
  - So, if necessary, negate constraints with negative right-hand sides.
- Then convert each constraint to an equation.
  - For a $\leq$ constraint, add a slack variable to LHS.
  - For a $\geq$ constraint, subtract a surplus variable and add an artificial variable to LHS.
  - For an $=$ constraint, add an artificial variable to LHS.

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The two-phase simplex algorithm.

[P&S §28]

<u>Problem:</u> If we have artificial variables with nonzero values, then the solution is <u>not feasible</u> in the original LP.

But if we could find a way to make all artificial variables zero, then we would have a feasible solution to original LP.

Here's the trick:

We define $\xi =$ (sum of artificial vars).

Then <u>minimize</u> ξ (subject to LP constraints).

- If we can make $\xi = 0$, then all artificial variables have the value zero, and we have a feasible solution to the original LP.

- If we can't make $\xi = 0$, then there is no feasible solution in which all artificial variables are zero — so original LP is <u>infeasible</u>.

How do we minimize ξ?

With the simplex algorithm!

Minimize ξ by <u>maximizing</u> $-\xi$.

We call $-\xi$ the <u>artificial objective function.</u>

- However, so that our artificial variable columns will be columns of the identity matrix, we first need to rewrite $-\xi$ in terms of non-artificial variables.

<u>Example.</u>

$$\begin{array}{lrcrcrcrcrcrcl}
\max & 7x_1 & + & 5x_2 & & & & & & & & & & \\
\text{s.t.} & 3x_1 & + & 2x_2 & + & s_1 & & & & & & & = & 10 \\
 & x_1 & + & x_2 & & & - & p_2 & + & a_2 & & & = & 4 \\
 & -x_1 & + & x_2 & & & & & & & + & a_3 & = & 3
\end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ p_2 \geq 0,\ a_2 \geq 0,\ a_3 \geq 0.$$

Artificial objective function: $-\xi = -a_2 - a_3$

From second constraint: $a_2 = 4 - x_1 - x_2 + p_2$

From third constraint: $a_3 = 3 + x_1 - x_2$

So $-\xi = -(4 - x_1 - x_2 + p_2) - (3 + x_1 - x_2)$

$= -7 + 2x_2 - p_2.$

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Two-phase — 2

Therefore: $-2x_2 + p_2 + (-\xi) = -7$.

We'll add a row for this equation to the simplex tableau.

Note minus sign

	x_1	x_2	s_1	p_2	a_2	a_3	z	$-\xi$	RHS
objective row →	-7	-5	0	0	0	0	1	0	0
artificial objective row →	0	-2	0	1	0	0	0	1	-7
	3	2	1	0	0	0	0	0	10
	1	1	0	-1	1	0	0	0	4
	-1	1	0	0	0	1	0	0	3

Now we apply the two-phase simplex algorithm:

Phase I: Maximize $-\xi$.
- Use the artificial objective row to choose pivot columns.
- Ignore the objective row, except still apply row operations to it during pivots.
- Done when the artificial objective row has no negative entries (except perhaps in RHS column).

If the optimal value of $-\xi$ is 0, then continue to Phase II: Maximize z.
- Cross out artificial objective row and artificial variable columns — don't need them any more. Apply simplex algo to remaining tableau.

Example (continued).

Initial (two-phase) simplex tableau:

x_1	x_2	s_1	p_2	a_2	a_3	z	$-\xi$	RHS	(Test ratio)
−7	−5	0	0	0	0	1	0	0	
0	−2	0	1	0	0	0	1	−7	
3	2	1	0	0	0	0	0	10	(5)
1	1	0	−1	1	0	0	0	4	(4)
−1	1	0	0	0	1	0	0	3	(3)

Pivot column (because negative entry in artificial objective row).

Pivot entry (minimum test ratio).

After pivot:

x_1	x_2	s_1	p_2	a_2	a_3	z	$-\xi$	RHS	(Test ratio)
−12	0	0	0	0	5	1	0	15	
−2	0	0	1	0	2	0	1	−1	
5	0	1	0	0	−2	0	0	4	(4/5)
2	0	0	−1	1	−1	0	0	1	(1/2)
−1	1	0	0	0	1	0	0	3	—

Pivot column (negative entry in artificial objective row).

Pivot entry (minimum test ratio).

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Two-phase – 3

After another pivot:

x_1	x_2	s_1	p_2	a_2	a_3	z	$-\xi$	RHS
0	0	0	−6	6	−1	1	0	21
0	0	0	0	1	1	0	1	0
0	0	1	5/2	−5/2	1/2	0	0	3/2
1	0	0	−1/2	1/2	−1/2	0	0	1/2
0	1	0	−1/2	1/2	1/2	0	0	7/2

No negative entries in artificial objective row to the left of RHS column, so **done** with Phase I. ($-\xi$ is maximized.)

Look at value of $-\xi$ now.
We have $-\xi = 0$. Since ξ is the sum of all artificial variables, and artificial variables (like all variables in the simplex algorithm) are nonnegative, this means that all artificial variables now have the value 0 (which they do, because they are nonbasic).

So Phase I was successful — we found a basic feasible solution to the original LP.

Go on to Phase II.

Phase II. We don't need the artificial objective row or the artificial variable columns any more (or the column for $-\xi$), so we can just get rid of them:

x_1	x_2	s_1	p_2	z	RHS
0	0	0	-6	1	21
0	0	1	$5/2$	0	$3/2$
1	0	0	$-1/2$	0	$1/2$
0	1	0	$-1/2$	0	$7/2$

Pivot column (negative entry in objective row).

Pivot entry (the only possibility — both other entries are negative).

Continue with simplex algorithm, now using the real objective row:

After pivot:

x_1	x_2	s_1	p_2	z	RHS
0	0	2.4	0	1	24.6
0	0	0.4	1	0	0.6
1	0	0.2	0	0	0.8
0	1	0.2	0	0	3.8

No negative entries in objective row $\Rightarrow$ OPTIMAL.

Optimal solution: $x_1 = 0.8$, $x_2 = 3.8$, $s_1 = 0$, $p_2 = 0.6$.

Optimal objective value: $z = 24.6$.