Why did I choose the basis $\{w, s_1, s_3\}$ in this example? What if I had chosen $\{w, s_1, s_2\}$ instead? ↓ 2 June

To find the corresponding basic solution, we set $p=0$ and $s_3=0$ and solve the resulting system via Gauss-Jordan elimination. The s_1 and s_2 columns are already columns of the identity matrix, so we are done if we make the w column be $[0,0,1]^T$:

p	w	s_1	s_2	s_3	RHS
1	1	1	0	0	100
1	4	0	1	0	160
10	20	0	0	1	1100

↓ Multiply row 3 by 1/20

p	w	s_1	s_2	s_3	RHS
1	1	1	0	0	100
1	4	0	1	0	160
1/2	1	0	0	1/20	55

↓ Subtract row 3 from row 1
Subtract 4(row 3) from row 2

p	w	s_1	s_2	s_3	RHS
1/2	0	1	0	−1/20	45
−1	0	0	1	−1/5	−60
1/2	1	0	0	1/20	55

So the corresponding basic solution is $p=0$, $w=55$, $s_1=45$, $s_2=-60$, $s_3=0$.

2 June

Simplex tableau and pivoting — 3

Note that this basic solution is <u>NOT</u> feasible, because s_2 is negative. What happened? And how can we detect whether this will happen <u>before</u> we do the pivot?

- First, notice that the entries in the RHS column are always the values of the basic variables (and the nonbasic variables are zero). So a basic solution is feasible if and only if <u>all of the entries in the RHS column are nonnegative.</u>

- Now consider a simplified tableau:

x	RHS
a	b
c	d

Suppose we pivot on the entry a:

1. Multiply row 1 by $1/a$ (so we need <u>$a \neq 0$</u>):

x	RHS
1	b/a
c	d

2. Subtract $c \cdot$(row 1) from row 2:

x	RHS
1	b/a
0	$d - c(b/a)$

(Note that entries in RHS column after the pivot depend only on entries in pivot column and RHS column before pivot.)

— In order for the corresponding basic solution to be feasible, we need

$$\frac{b}{a} \geq 0 \quad \text{and} \quad d - c\left(\frac{b}{a}\right) \geq 0.$$

— Assuming that we began with a tableau corresponding to a bfs, we have $b \geq 0$ and $d \geq 0$. Suppose for the moment that we have $b > 0$ and $d > 0$ (non-degenerate). Then we need $a > 0$ so that $b/a \geq 0$. So, first rule:

$$\boxed{\text{Pivot only on positive entries.}}$$

— Assuming that $b/a \geq 0$, then:

- If $c \leq 0$, we have $d - c(b/a) \geq d \geq 0$. So, if we follow the first rule, nothing can go wrong in rows with nonpositive entries in the pivot column.

2 June

Simplex tableau and pivoting — 4

- If $c > 0$, then

$$d - c\left(\frac{b}{a}\right) \geq 0 \iff \frac{d}{c} \geq \frac{b}{a}.$$

Therefore, in order to get all nonnegative entries in the RHS column after the pivot, we need the quotient b/a in the pivot <u>row</u> to be the <u>minimum</u> of all such quotients in rows having positive entries in the pivot <u>column</u>.

<u>Defn.</u> The <u>test ratio</u> for an entry a in a simplex tableau is b/a, where b is the entry in the RHS column in the same row as a.

Second rule (refinement of the first):

> Among all <u>positive</u> entries in the pivot column, pivot on the one having the <u>minimum</u> test ratio.

(Ties can be broken arbitrarily. But doing so can result in <u>cycling</u>. There are rules for breaking ties to avoid this, such as Bland's rule. See Section 2.7 of P&S for a discussion of these issues.)

Overview of pivoting

- Choose a nonbasic variable to bring into the basis. This determines the pivot column.

- For each positive entry in the pivot column, compute the test ratio.

- Pivot on the positive entry having the minimum test ratio.

  (This means:
  1. Divide the pivot row by the pivot entry.
  2. Add or subtract a multiple of the new pivot row from each other row in order to make all other entries in the pivot column equal to zero.)

Annoying special cases.

- Ties in test ratio: see bottom of previous page.
- No positive entry in pivot column: can't pivot there. In the full simplex algorithm, this indicates LP is unbounded.
- Degeneracy (some entry in RHS column is zero). The same pivoting rules apply, but the justifications are more subtle. See P&S.

2 June

Objective row of simplex tableau.

Example. (Still Farmer Brown.)

Call the objective value z.

$$\max\ z = 40p + 120w$$

$$\begin{aligned}
\text{s.t.}\quad p + w + s_1 &= 100\\
p + 4w + s_2 &= 160\\
10p + 20w + s_3 &= 1100
\end{aligned}$$

$$p \geq 0,\ w \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0.$$

Rewrite the equation for z like this:

$$40p + 120w - z = 0$$

Note: P&S do not make this flip.

Or, to make the coefficient of z positive:

$$-40p - 120w + z = 0$$

Include a row for this equation at the top of the simplex tableau. This is called the objective row.

p	w	s_1	s_2	s_3	z	RHS
-40	-120	0	0	0	1	0
1	1	1	0	0	0	100
1	4	0	1	0	0	160
10	20	0	0	1	0	1100

Note: Because nonbasic vars have the value 0, this entry is the value of z.

Meanings of the numbers in the objective row.

- Basic columns always have 0 in the objective row (after the appropriate pivots).

- Nonbasic variables have the value 0.

  - But if the corresponding entry in the objective row is <u>negative</u>, then increasing the value of that variable to a positive value will <u>increase</u> the value of z.

    (Because in the equation expressing z as a function of the nonbasic variables, that variable has a <u>positive</u> coefficient.)

- So: <u>Improve the objective value by</u>

  > pivoting in a (nonbasic) column that has a negative entry in the objective row.

- <u>Common rule of thumb: Pivot in the column that has the</u> <u>most negative</u> entry in the objective row.

2 June

Objective row — 2

Example. Farmer Brown.

Initial simplex tableau:

p	w	s_1	s_2	s_3	z	RHS
-40	-120	0	0	0	1	0
1	1	1	0	0	0	100
1	4	0	1	0	0	160
10	20	0	0	1	0	1100

↑

The w column has the most negative entry in the objective row, so this will be our pivot column.

[Why? Objective row here means

$$-40p - 120w + z = 0,$$

So $z = 40p + 120w$, so if we give w a positive value (by bringing it into the basis), then we will increase the value of z.]

Test ratios:
(not for objective row, because we want z always a basic var)

First row: $100/1 = 100$
Second row: $160/4 = 40$ ← smallest.
Third row: $1100/20 = 55$

Pivot:

p	w	s_1	s_2	s_3	z	RHS
-40	-120	0	0	0	1	0
1	1	1	0	0	0	100
1	4	0	1	0	0	160
10	20	0	0	1	0	1100

$\downarrow$ Multiply row 2 by 1/4

p	w	s_1	s_2	s_3	z	RHS
-40	-120	0	0	0	1	0
1	1	1	0	0	0	100
1/4	1	0	1/4	0	0	40
10	20	0	0	1	0	1100

$\downarrow$ Add 120(row 2) to objective row
Subtract row 2 from row 1
Subtract 20(row 2) from row 3

p	w	s_1	s_2	s_3	z	RHS
-10	0	0	30	0	1	4800
3/4	0	1	$-1/4$	0	0	60
1/4	1	0	1/4	0	0	40
5	0	0	-5	1	0	300

New basic solution: $p=0$, $w=40$, $s_1=60$, $s_2=30$, $s_3=300$.

Objective row here means $-10p+30s_2+z=4800$,
so $z=4800+10p-30s_2$. $(*)$

- $p=s_2=0$, so current value of z is 4800.
- Note that p has a positive coefficient in $(*)$, negative entry in obj. row, so increasing p will improve z.

2 June

Objective row — 3

Next pivot:

↓ Pivot column.

p	w	s_1	s_2	s_3	z	RHS	Test ratios:
-10	0	0	30	0	1	4800	
$3/4$	0	1	$-1/4$	0	0	60	(80)
$1/4$	1	0	$1/4$	0	0	40	(160)
5	0	0	-5	1	0	300	(60)

↓ Multiply row 3 by 1/5

p	w	s_1	s_2	s_3	z	RHS
-10	0	0	30	0	1	4800
$3/4$	0	1	$-1/4$	0	0	60
$1/4$	1	0	$1/4$	0	0	40
1	0	0	-1	$1/5$	0	60

↓ Add 10(row 3) to objective row
Subtract (3/4)(row 3) from row 1
Subtract (1/4)(row 3) from row 2

p	w	s_1	s_2	s_3	z	RHS
0	0	0	20	2	1	5400
0	0	1	$1/2$	$-3/20$	0	15
0	1	0	$1/2$	$-1/20$	0	25
1	0	0	-1	$1/5$	0	60

New basic solution: $p=60$, $w=25$, $s_1=15$, $s_2=0$, $s_3=0$.

Objective row: $20s_2 + 2s_3 + z = 5400$

$\Rightarrow z = 5400 - 20s_2 - 2s_3$.

So
- current value of z is 5400;
- since $s_2 \geq 0$ and $s_3 \geq 0$, this is <u>optimal</u>!

Essential outline of the SIMPLEX ALGORITHM.

1. Rewrite LP as a maximization problem in standard form.

2. Write the initial simplex tableau.
   (Don't forget to negate coeffs in objective row.)

3. For as long as there exists a (nonbasic) column with a negative entry in the objective row:

   (a) Choose such a column as the pivot column.

   (b) Compute test ratios for rows having positive entries in that column.

   (c) Choose pivot row with minimum test ratio. Pivot on the corresponding entry.

4. When all entries in the objective row are nonnegative, corresponding bfs is *optimal*.

2 June Objective row and simplex algorithm — 4

Potential complications ignored in the preceding outline:

- Initial simplex tableau may not represent a basic feasible solution.

- If more than one column has a negative entry in the objective row, which is the best to pick?

- If there is a tie for minimum test ratio, which row is best to pick? [See P&S §2.7.]

- If pivot column has no positive entries, can't pivot there.
  - Implies LP is unbounded. (Why?)

- Effects of degeneracy.

- Detection of alternative optimal solutions (i.e., whether an optimal solution is unique).
  - Optimal solution is unique if all nonbasic columns have positive (i.e., nonzero) entries in the objective row.