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Combinatorial optimization overview

- Combinatorial optimization is the study of problems that involve a search for the "best" option among a (usually finite) set of choices.

- Example: Factory placement.
  A company has ten possible sites at which it can build factories.
  A factory can produce 100 units/month of output. Each site has a different building cost:

Site:	1	2	3	4	...
Building cost:	\$1.2M	\$1.8 M	\$0.8 M	\$2.1M	...

The company has a list of customers with monthly demands:

Customer:	A	B	C	D	E	...
Monthly demand:	12	16	23	7	10	...

Shipping costs per unit:

TO \ FROM	1	2	3	...
A	\$4	\$6	\$13	...
B	10	8	9	...
C	7	5	3	...
⋮	⋮	⋮	⋮	⋱

Where should factories be built to satisfy customer demand over 5 years at min cost?

- Example: Bin packing.
  - List of items of various sizes:

Item:	1	2	3	4	...
Size:	3	7	12	9	...

  - Supply of bins, each having the same capacity (say, capacity 50).
  - How to place all items into bins, using the minimum number ~~of~~ bins, without exceeding bin capacity?
  - "Steel mill slab problem": Each item has a <u>color</u>, and the number of different colors in each bin can be no more than 2.

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- Example: Swim meets. Two teams.
  - Each team can enter at most three swimmers in any one individual (non-relay) event.
  - Each team can provide at most three "entries" in each relay event, where an entry is a group of four swimmers.
  - Each swimmer can enter at most four events.
  - Each swimmer can enter at most two individual events.
  - In a relay event, a team cannot be awarded points for more than two finishing places.
  - Coach has information about event times of swimmers, and limited information about opposing team.
  - How to assign swimmers to events to maximize probability of winning?

[Nowak, Epelman, Pollock]

More examples:

- Project scheduling (CPM)
- Minimum spanning tree
- Traveling salesman

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Linear programming — intro example.

- Farmer Brown is planning to raise potatoes and wheat.
- 100 acres of land available.
- 160 days of labor available.
- One acre of wheat will require 4 days of labor.
- One acre of potatoes will require 1 day of labor.
- \$1100 available for start-up costs of planting and cultivating.
- \$10 per acre to plant/cultivate potatoes.
- \$20 per acre to plant/cultivate wheat.
- Expected revenue:
  \$40/acre for potatoes,
  \$120/acre for wheat.

- How many acres of each crop should be planted to maximize total revenue?

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LP formulation.

1. Identify <u>variables</u> and their <u>domains</u>.

- Variables usually represent <u>decisions</u> to be made — quantities under your control.

- Imagine a form letter describing the best course of action to take, the best solution. Write a sentence that could be used to describe <u>any</u> solution, using blanks in place of numbers. Those blanks are the variables.

- Sometimes auxiliary variables can be helpful — this usually becomes evident when trying to formulate constraints or objective function.

- Each variable <u>must</u> have a corresponding domain. In an LP, there are three possible domains:

$$\underbrace{x \geq 0,}_{\text{By far the most common.}} \qquad x \leq 0, \qquad x \text{ unrestricted.}$$

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LP formulation.

2. Identify the <u>objective</u>.

   - What quantity are you aiming to maximize or minimize?

3. Write the <u>objective function</u> in terms of the defined variables.

   - Remember that the objective function must be <u>linear</u>.

4. Identify the <u>constraints</u> in the problem and express each one in terms of the defined variables.

   - Remember that every constraint must be a <u>linear</u> inequality (or equality). <u>Inequality</u> cannot be <u>strict</u>.

   - Linear means:
     - No raising variables to powers,
     - No multiplication of variables together,
     - No dividing by variables,
     - No functions like sin(), $\sqrt{\cdot}$, etc. applied to variables.

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LP formulation for Farmer Brown example.

Variables:

p: number of acres of potatoes
w: number of acres of wheat

Domains: $p \geq 0, \quad w \geq 0$

Objective: Maximize revenue.

Objective function: $40p + 120w$

Constraints:

Land (resource): $p + w \leq 100$

Labor (resource): $p + 4w \leq 160$

Capital (resource): $10p + 20w \leq 1100$

Written:

$$
\begin{aligned}
\text{maximize} \quad & 40p + 120w \\
\text{subject to} \quad & p + w \leq 100 \quad \text{[land]} \\
& p + 4w \leq 160 \quad \text{[labor]} \\
& 10p + 20w \leq 1100 \quad \text{[capital]} \\
& p \geq 0, \quad w \geq 0.
\end{aligned}
$$

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"Anatomy" of an LP.

Objective $\rightarrow$ $\boxed{\max \ \boxed{40p + 120w}}$ $\leftarrow$ Objective function

s.t.

Constraints $\rightarrow$
$$\boxed{\begin{array}{rcrcl} p & + & w & \leq & 100 \\ p & + & 4w & \leq & 160 \\ 10p & + & 20w & \leq & 1100 \end{array}}$$

Variable domains $\rightarrow$ $\boxed{p \geq 0, \quad w \geq 0}$

Terminology

- <u>Solution</u>: An assignment of values to variables.
- <u>Feasible solution</u>: A solution that satisfies all constraints (and domains).
- <u>Feasible region</u> (a.k.a. feasible set): The set of all feasible solutions.
- <u>Objective value</u>: The value of the objective function corresponding to a given solution.
- <u>Optimal (feasible) solution</u>: A feasible solution whose objective value is at least as "good" as that of any other feasible solution.
- <u>Optimal objective value</u>: The objective value of an optimal feasible solution.

More terminology.

- Feasible LP: An LP with at least one feasible solution.
- Infeasible LP: One that is not feasible.
- Unbounded LP: A feasible LP with no optimal feasible solution.

Solving an LP means attempting to find an optimal feasible solution (not just the optimal objective value!).

Possible outcomes:

- LP is infeasible.
- LP is feasible.
  - LP is unbounded.
  - LP has an optimal feasible solution.
    - LP has a unique optimal feasible solution.
    - LP has a nonunique optimal feasible solution (i.e., at least two).

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Solving an LP with Maple.

Farmer Brown:

$$\begin{array}{llcl} \max & 40p + 120w & & \\ \text{s.t.} & p + w & \leq 100 & \text{[land]} \\ & p + 4w & \leq 160 & \text{[labor]} \\ & 10p + 20w & \leq 1100 & \text{[capital]} \end{array}$$

$$p \geq 0, \quad w \geq 0.$$

In Maple:

```
restart;

with(Optimization);

f:= (p,w) -> 40*p + 120*w;

Constraints := [ p+w<=100, p+4*w<=160,
                 10*p+20*w<=1100 ];

LPSolve( f(p,w), constraints, 'maximize',
         assume=nonnegative );
```

Output: [5400., [p=60., w=25.]]