Combinatorial Optimization

Midterm examination: solutions

1. Solve the following linear program. You may not use Maple for this problem.

$$\begin{array}{rl}
\text{minimize} & 3x_1 + 2x_2 - x_3 - 4x_4 \\
\text{subject to} & x_1 + x_2 + x_3 + x_4 = 10 \\
& 2x_1 + x_3 - 2x_4 \geq 6 \\
& x_1 + 3x_4 \leq 30 \\
& 3x_1 + x_2 \geq -8 \\
& x_2 + 3x_3 \geq 3 \\
\multicolumn{2}{c}{x_1 \geq 0, \quad x_2 \text{ unrestricted}, \quad x_3 \leq 0, \quad x_4 \geq 0.}
\end{array}$$

▷ **Solution.** There are several possible approaches. One approach begins by writing the dual:

$$\begin{array}{rl}
\text{maximize} & 10y_1 + 6y_2 + 30y_3 - 8y_4 + 3y_5 \\
\text{subject to} & y_1 + 2y_2 + y_3 + 3y_4 \leq 3 \\
& y_1 + y_4 + y_5 = 2 \\
& y_1 + y_2 + 3y_5 \geq -1 \\
& y_1 - 2y_2 + 3y_3 \leq -4 \\
\multicolumn{2}{c}{y_1 \text{ unrestricted}, \ y_2 \geq 0, \ y_3 \leq 0, \ y_4 \geq 0, \ y_5 \geq 0.}
\end{array}$$

In order to make the right-hand sides nonnegative, we multiply the last two constraints by -1. Then we convert to standard variable domains by making the substitutions $y_1 = y_1^+ - y_1^-$ and $y_3 = -\bar{y}_3$ with $y_1^+ \geq 0$, $y_1^- \geq 0$, and $\bar{y}_3 \geq 0$:

$$\begin{array}{rl}
\text{maximize} & 10y_1^+ - 10y_1^- + 6y_2 - 30\bar{y}_3 - 8y_4 + 3y_5 \\
\text{subject to} & y_1^+ - y_1^- + 2y_2 - \bar{y}_3 + 3y_4 \leq 3 \\
& y_1^+ - y_1^- + y_4 + y_5 = 2 \\
& -y_1^+ + y_1^- - y_2 - 3y_5 \leq 1 \\
& -y_1^+ + y_1^- + 2y_2 + 3\bar{y}_3 \geq 4 \\
\multicolumn{2}{c}{y_1^+ \geq 0, \ y_1^- \geq 0, \ y_2 \geq 0, \ \bar{y}_3 \geq 0, \ y_4 \geq 0, \ y_5 \geq 0.}
\end{array}$$

Next we insert slack, surplus, and artificial variables:

$$\begin{array}{rl}
\text{maximize} & 10y_1^+ - 10y_1^- + 6y_2 - 30\bar{y}_3 - 8y_4 + 3y_5 \\
\text{subject to} & y_1^+ - y_1^- + 2y_2 - \bar{y}_3 + 3y_4 + s_1 = 3 \\
& y_1^+ - y_1^- + y_4 + y_5 + a_2 = 2 \\
& -y_1^+ + y_1^- - y_2 - 3y_5 + s_3 = 1 \\
& -y_1^+ + y_1^- + 2y_2 + 3\bar{y}_3 - p_4 + a_4 = 4 \\
\multicolumn{2}{c}{y_1^+ \geq 0, \ y_1^- \geq 0, \ y_2 \geq 0, \ \bar{y}_3 \geq 0, \ y_4 \geq 0, \ y_5 \geq 0, \ s_1 \geq 0, \ a_2 \geq 0, \ s_3 \geq 0, \ p_4 \geq 0, \ a_4 \geq 0.}
\end{array}$$

We form the initial tableau for the two-phase simplex algorithm:

y_1^+	y_1^-	y_2	$\bar{y}_3$	y_4	y_5	s_1	a_2	s_3	p_4	a_4	z	$-\xi$	RHS
-10	10	-6	30	8	-3	0	0	0	0	0	1	0	0
0	0	-2	-3	-1	-1	0	0	0	1	0	0	1	-6
1	-1	2	-1	3	0	1	0	0	0	0	0	0	3
1	-1	0	0	1	1	0	1	0	0	0	0	0	2
-1	1	-1	0	0	-3	0	0	1	0	0	0	0	1
-1	1	2	$\boxed{3}$	0	0	0	0	0	-1	1	0	0	4

Pivot on the 3 in the $\bar{y}_3$ column:

y_1^+	y_1^-	y_2	$\bar{y}_3$	y_4	y_5	s_1	a_2	s_3	p_4	a_4	z	$-\xi$	RHS
0	0	−26	0	8	−3	0	0	0	10	−10	1	0	−40
−1	1	0	0	−1	−1	0	0	0	0	1	0	1	−2
2/3	−2/3	8/3	0	3	0	1	0	0	−1/3	1/3	0	0	13/3
1	−1	0	0	1	$\boxed{1}$	0	1	0	0	0	0	0	2
−1	1	−1	0	0	−3	0	0	1	0	0	0	0	1
−1/3	1/3	2/3	1	0	0	0	0	0	−1/3	1/3	0	0	4/3

Pivot on the 1 in the y_5 column:

y_1^+	y_1^-	y_2	$\bar{y}_3$	y_4	y_5	s_1	a_2	s_3	p_4	a_4	z	$-\xi$	RHS
3	−3	−26	0	11	0	0	3	0	10	−10	1	0	−34
0	0	0	0	0	0	0	1	0	0	1	0	1	0
2/3	−2/3	8/3	0	3	0	1	0	0	−1/3	1/3	0	0	13/3
1	−1	0	0	1	1	0	1	0	0	0	0	0	2
2	−2	−1	0	3	0	0	3	1	0	0	0	0	7
−1/3	1/3	2/3	1	0	0	0	0	0	−1/3	1/3	0	0	4/3

All entries in the artificial objective row are nonnegative, so Phase I is complete. The artificial objective value is 0, so Phase I was successful. All artificial variables are nonbasic, so we can continue to Phase II. We delete the artificial objective row and the $-\xi$ column, but we will keep the artificial variable columns because we want the optimal solution to the minimizing linear program (we are applying the simplex algorithm to the dual).

y_1^+	y_1^-	y_2	$\bar{y}_3$	y_4	y_5	s_1	a_2	s_3	p_4	a_4	z	RHS
3	−3	−26	0	11	0	0	3	0	10	−10	1	−34
2/3	−2/3	$\boxed{8/3}$	0	3	0	1	0	0	−1/3	1/3	0	13/3
1	−1	0	0	1	1	0	1	0	0	0	0	2
2	−2	−1	0	3	0	0	3	1	0	0	0	7
−1/3	1/3	2/3	1	0	0	0	0	0	−1/3	1/3	0	4/3

Pivot on the 8/3 in the y_2 column:

y_1^+	y_1^-	y_2	$\bar{y}_3$	y_4	y_5	s_1	a_2	s_3	p_4	a_4	z	RHS
19/2	−19/2	0	0	161/4	0	39/4	3	0	27/4	−27/4	1	33/4
1/4	−1/4	1	0	9/8	0	3/8	0	0	−1/8	1/8	0	13/8
1	−1	0	0	1	1	0	1	0	0	0	0	2
9/4	−9/4	0	0	33/8	0	3/8	3	1	−1/8	1/8	0	69/8
−1/2	$\boxed{1/2}$	0	1	−3/4	0	−1/4	0	0	−1/4	1/4	0	1/4

Pivot on the 1/2 in the y_1^- column:

y_1^+	y_1^-	y_2	$\bar{y}_3$	y_4	y_5	s_1	a_2	s_3	p_4	a_4	z	RHS
0	0	0	19	26	0	5	3	0	2	−2	1	13
0	0	1	1/2	3/4	0	1/4	0	0	−1/4	1/4	0	7/4
0	0	0	2	−1/2	1	−1/2	1	0	−1/2	1/2	0	5/2
0	0	0	9/2	3/4	0	−3/4	3	1	−5/4	5/4	0	39/4
−1	1	0	2	−3/2	0	−1/2	0	0	−1/2	1/2	0	1/2

All entries in the objective row are nonnegative, so this tableau is optimal. But this is the optimal tableau for the *dual*, so the optimal solution to the *primal* is found in the objective row in columns corresponding to slack and artificial variables (i.e., s_1, a_2, s_3, and a_4)—except that we negated the third and fourth constraints in the dual, so we need to negate the values at the top of the s_3 and a_4 columns. Therefore, the optimal solution to the primal is $x_1 = 5$, $x_2 = 3$, $x_3 = 0$, $x_4 = 2$, which yields the objective value 13.

Another approach is to negate the objective function in the original linear program and maximize. Multiply the fourth constraint by -1 in order to make the right-hand side nonnegative, and then convert to standard variable domains by making the substitutions $x_2 = x_2^+ - x_2^-$ and $x_3 = -\bar{x}_3$ with $x_2^+ \geq 0$, $x_2^- \geq 0$, and $\bar{x}_3 \geq 0$. The result is the following linear program.

$$\begin{array}{rl} \text{maximize} & -3x_1 - 2x_2^+ + 2x_2^- - \bar{x}_3 + 4x_4 \\ \text{subject to} & x_1 + x_2^+ - x_2^- - \bar{x}_3 + x_4 = 10 \\ & 2x_1 - \bar{x}_3 - 2x_4 \geq 6 \\ & x_1 + 3x_4 \leq 30 \\ & -3x_1 - x_2^+ + x_2^- \leq 8 \\ & x_2^+ - x_2^- - 3\bar{x}_3 \geq 3 \\ & x_1 \geq 0,\ x_2^+ \geq 0,\ x_2^- \geq 0,\ \bar{x}_3 \geq 0,\ x_4 \geq 0. \end{array} \tag{1}$$

Insert slack, surplus, and artificial variables:

$$\begin{array}{rl} \text{maximize} & -3x_1 - 2x_2^+ + 2x_2^- - \bar{x}_3 + 4x_4 \\ \text{subject to} & x_1 + x_2^+ - x_2^- - \bar{x}_3 + x_4 + a_1 = 10 \\ & 2x_1 - \bar{x}_3 - 2x_4 - p_2 + a_2 = 6 \\ & x_1 + 3x_4 + s_3 = 30 \\ & -3x_1 - x_2^+ + x_2^- + s_4 = 8 \\ & x_2^+ - x_2^- - 3\bar{x}_3 - p_5 + a_5 = 3 \end{array}$$

$$x_1 \geq 0,\ x_2^+ \geq 0,\ x_2^- \geq 0,\ \bar{x}_3 \geq 0,\ x_4 \geq 0,\ a_1 \geq 0,\ p_2 \geq 0,\ a_2 \geq 0,\ s_3 \geq 0,\ s_4 \geq 0,\ p_5 \geq 0,\ a_5 \geq 0.$$

Next we form the initial tableau for the two-phase simplex algorithm:

x_1	x_2^+	x_2^-	$\bar{x}_3$	x_4	a_1	p_2	a_2	s_3	s_4	p_5	a_5	z	$-\xi$	RHS
3	2	−2	1	−4	0	0	0	0	0	0	0	1	0	0
−3	−2	2	5	1	0	1	0	0	0	1	0	0	1	−19
1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	10
$\boxed{2}$	0	0	−1	−2	0	−1	1	0	0	0	0	0	0	6
1	0	0	0	3	0	0	0	1	0	0	0	0	0	30
−3	−1	1	0	0	0	0	0	0	1	0	0	0	0	8
0	1	−1	−3	0	0	0	0	0	0	−1	1	0	0	3

Pivot on the 2 in the x_1 column:

x_1	x_2^+	x_2^-	$\bar{x}_3$	x_4	a_1	p_2	a_2	s_3	s_4	p_5	a_5	z	$-\xi$	RHS
0	2	−2	5/2	−1	0	3/2	−3/2	0	0	0	0	1	0	−9
0	−2	2	7/2	−2	0	−1/2	3/2	0	0	1	0	0	1	−10
0	1	−1	−1/2	2	1	1/2	−1/2	0	0	0	0	0	0	7
1	0	0	−1/2	−1	0	−1/2	1/2	0	0	0	0	0	0	3
0	0	0	1/2	4	0	1/2	−1/2	1	0	0	0	0	0	27
0	−1	1	−3/2	−3	0	−3/2	3/2	0	1	0	0	0	0	17
0	$\boxed{1}$	−1	−3	0	0	0	0	0	0	−1	1	0	0	3

Pivot on the 1 in the bottom row in the x_2^+ column:

x_1	x_2^+	x_2^-	$\bar{x}_3$	x_4	a_1	p_2	a_2	s_3	s_4	p_5	a_5	z	$-\xi$	RHS
0	0	0	$17/2$	-1	0	$3/2$	$-3/2$	0	0	2	-2	1	0	-15
0	0	0	$-5/2$	-2	0	$-1/2$	$3/2$	0	0	-1	2	0	1	-4
0	0	0	$\boxed{5/2}$	2	1	$1/2$	$-1/2$	0	0	1	-1	0	0	4
1	0	0	$-1/2$	-1	0	$-1/2$	$1/2$	0	0	0	0	0	0	3
0	0	0	$1/2$	4	0	$1/2$	$-1/2$	1	0	0	0	0	0	27
0	0	0	$-9/2$	-3	0	$-3/2$	$3/2$	0	1	-1	1	0	0	20
0	1	-1	-3	0	0	0	0	0	0	-1	1	0	0	3

Pivot on the 5/2 in the $\bar{x}_3$ column:

x_1	x_2^+	x_2^-	$\bar{x}_3$	x_4	a_1	p_2	a_2	s_3	s_4	p_5	a_5	z	$-\xi$	RHS
0	0	0	0	$-39/5$	$-17/5$	$-1/5$	$1/5$	0	0	$-7/5$	$7/5$	1	0	$-143/5$
0	0	0	0	0	1	0	1	0	0	0	1	0	1	0
0	0	0	1	$4/5$	$2/5$	$1/5$	$-1/5$	0	0	$2/5$	$-2/5$	0	0	$8/5$
1	0	0	0	$-3/5$	$1/5$	$-2/5$	$2/5$	0	0	$1/5$	$-1/5$	0	0	$19/5$
0	0	0	0	$18/5$	$-1/5$	$2/5$	$-2/5$	1	0	$-1/5$	$1/5$	0	0	$131/5$
0	0	0	0	$3/5$	$9/5$	$-3/5$	$3/5$	0	1	$4/5$	$-4/5$	0	0	$136/5$
0	1	-1	0	$12/5$	$6/5$	$3/5$	$-3/5$	0	0	$1/5$	$-1/5$	0	0	$39/5$

All entries in the artificial objective row are nonnegative, so Phase I is complete. The artificial objective value is 0, so Phase I was successful. All artificial variables are nonbasic, so we can continue to Phase II. We delete the artificial objective row, the artificial variable columns, and the $-\xi$ column.

x_1	x_2^+	x_2^-	$\bar{x}_3$	x_4	p_2	s_3	s_4	p_5	z	RHS
0	0	0	0	$-39/5$	$-1/5$	0	0	$-7/5$	1	$-143/5$
0	0	0	1	$\boxed{4/5}$	$1/5$	0	0	$2/5$	0	$8/5$
1	0	0	0	$-3/5$	$-2/5$	0	0	$1/5$	0	$19/5$
0	0	0	0	$18/5$	$2/5$	1	0	$-1/5$	0	$131/5$
0	0	0	0	$3/5$	$-3/5$	0	1	$4/5$	0	$136/5$
0	1	-1	0	$12/5$	$3/5$	0	0	$1/5$	0	$39/5$

Pivot on the 4/5 in the x_4 column:

x_1	x_2^+	x_2^-	$\bar{x}_3$	x_4	p_2	s_3	s_4	p_5	z	RHS
0	0	0	$39/4$	0	$7/4$	0	0	$5/2$	1	-13
0	0	0	$5/4$	1	$1/4$	0	0	$1/2$	0	2
1	0	0	$3/4$	0	$-1/4$	0	0	$1/2$	0	5
0	0	0	$-9/2$	0	$-1/2$	1	0	-2	0	19
0	0	0	$-3/4$	0	$-3/4$	0	1	$1/2$	0	26
0	1	-1	-3	0	0	0	0	-1	0	3

All entries in the objective row are nonnegative, so this tableau is optimal. So the optimal solution to the linear program (1) is $x_1 = 5$, $x_2^+ = 3$, $x_2^- = 0$, $\bar{x}_3 = 0$, $x_4 = 2$, which yields the objective value -13. This means that the optimal solution to the original linear program is $x_1 = 5$, $x_2 = 3$, $x_3 = 0$, $x_4 = 2$, with objective value 13. □

2. Formulate a linear program for the following scenario. Then solve the linear program and interpret your results.

> Leisure Furniture, Inc. (LFI) makes outdoor chairs, tables, lounges, and benches. The main resources required for production are plastic webbing, metal tubing, and wood. The number of units of each resource required per item and the per-item profit are given below.
>
> | | Tubing
(feet) | Webbing
(yards) | Wood
(board feet) | Per-unit
profit |
> |---|---|---|---|---|
> | Chair | 18 | 28 | | \$11 |
> | Table | 18 | | 10 | \$18 |
> | Lounge | 24 | 35 | | \$14 |
> | Bench | 20 | | 14 | \$ 9 |
>
> The main activities in production are tube bending and assembly. The number of hours of each activity required per unit of each product are given in the table below.
>
> | | Chair | Table | Lounge | Bench |
> |---|---|---|---|---|
> | Tube bending | 1/3 | 1/8 | 1/4 | 1/8 |
> | Assembly | 1/2 | 1/2 | 1/3 | 1/3 |
>
> LFI is planning its first week's production for the spring. Available during that week are 120 hours of bending time, 160 hours of assembly time, 6,000 feet of tubing, 5,000 yards of webbing, and 1,000 board feet of wood. The lounge is the most popular product, so LFI wants to make at least 100 of them. Tables and chairs are commonly sold in sets of two chairs and a table, so they want to make at least twice as many chairs as tables. Otherwise they want to make the products that will maximize their profit if sold.

▷ **Solution.** Let C, T, L, and B be the numbers of chairs, tables, lounges, and benches to be made, respectively. All of these variables are nonnegative. The linear program below represents the problem of maximizing profit subject to the constraints in the problem.

$$
\begin{array}{rlr}
\text{maximize} & 11C + 18T + 14L + 9B & \text{[profit]} \\
\text{subject to} & 18C + 18T + 24L + 20B \le 6000 & \text{[tubing]} \\
 & 28C + 35L \le 5000 & \text{[webbing]} \\
 & 10T + 14B \le 1000 & \text{[wood]} \\
 & \frac{1}{3}C + \frac{1}{8}T + \frac{1}{4}L + \frac{1}{8}B \le 120 & \text{[bending]} \\
 & \frac{1}{2}C + \frac{1}{2}T + \frac{1}{3}L + \frac{1}{3}B \le 160 & \text{[assembly]} \\
 & L \ge 100 & \text{[lounge production]} \\
 & C - 2T \ge 0 & \text{[chairs \& tables]} \\
 & C \ge 0, \quad T \ge 0, \quad L \ge 0, \quad B \ge 0. &
\end{array}
$$

The following Maple worksheet solves this linear program.

```
> restart;
> with(Optimization);
```

```
[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve,
    QPSolve]
```

```
> profit:=(C,T,L,B)->11*C+18*T+14*L+9*B;
```

$$\mathit{profit} := (C, T, L, B) \to 11\,C + 18\,T + 14\,L + 9\,B$$

```
> constraints:=[18*C+18*T+24*L+20*B<=6000,28*C+35*L<=5000,
   10*T+14*B<=1000,C/3+T/8+L/4+B/8<=120,C/2+T/2+L/3+B/3<=160,
   L>=100,C-2*T>=0];
```

$$\mathit{constraints} := \Big[18\,C + 18\,T + 24\,L + 20\,B \leq 6000, 28\,C + 35\,L \leq 5000,$$
$$10\,T + 14\,B \leq 1000, \frac{1}{3}\,C + \frac{1}{8}\,T + \frac{1}{4}\,L + \frac{1}{8}\,B \leq 120, \frac{1}{2}\,C + \frac{1}{2}\,T + \frac{1}{3}\,L + \frac{1}{3}\,B \leq 160,$$
$$100 \leq L, 0 \leq C - 2\,T\Big]$$

```
> LPSolve(profit(C,T,L,B),constraints,'maximize',
   assume=nonnegative);
```

$$\big[2942.09183674067, [B = 52.2959183669781, C = 53.5714285714286, L = 100.,$$
$$T = 26.7857142862306]\big]$$

Unfortunately, the optimal solution to the linear program is not an integer solution, and a fractional solution doesn't really make sense in the context of the problem. (Sorry—I didn't intend for this to happen.) We can ask Maple to find the optimal *integer* solution by using `assume=nonnegint` instead of `assume=nonnegative` as an option to `LPSolve`. It turns out that we also have to increase the default depth limit for the search process. When we do this, we get the following result:

```
> LPSolve(profit(C,T,L,B),constraints,'maximize',
   assume=nonnegint,depthlimit=7);
```

$$\big[2922, [B = 52, C = 52, L = 101, T = 26]\big]$$

Hence, in order to maximize profit, LFI should produce 52 benches, 52 chairs, 101 lounges, and 26 tables, which will produce a profit of $2922. □

3. Find an extreme point of the feasible region of the following linear program.

$$\begin{array}{rl} \text{maximize} & x_1 - 2x_2 + x_3 \\ \text{subject to} & 3x_1 + 3x_2 - x_3 - 2x_4 \leq 90 \\ & x_1 + 5x_3 \geq 20 \\ & -4x_1 - x_2 + x_3 - x_4 = 6 \\ & x_2 - 2x_3 + x_4 \leq 40 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{array}$$

▷ **Solution.** An extreme point of the feasible region is a basic feasible solution, so that is what we are after in this problem. One approach is to perform Phase I of the two-phase simplex algorithm in order to get a basic feasible solution. We don't care about the objective function, so we don't need an objective row in our tableau (only the artificial objective row).

x_1	x_2	x_3	x_4	s_1	p_2	a_2	a_3	s_4	$-\xi$	RHS
3	1	−6	1	0	1	0	0	0	1	−26
3	3	−1	−2	1	0	0	0	0	0	90
1	0	$\boxed{5}$	0	0	−1	1	0	0	0	20
−4	−1	1	−1	0	0	0	1	0	0	6
0	1	−2	1	0	0	0	0	1	0	40

Pivot on the 5 in the x_3 column:

x_1	x_2	x_3	x_4	s_1	p_2	a_2	a_3	s_4	$-\xi$	RHS
$21/5$	1	0	1	0	$-1/5$	$6/5$	0	0	1	-2
$16/5$	3	0	-2	1	$-1/5$	$1/5$	0	0	0	94
$1/5$	0	1	0	0	$-1/5$	$1/5$	0	0	0	4
$-21/5$	-1	0	-1	0	$\boxed{1/5}$	$-1/5$	1	0	0	2
$2/5$	1	0	1	0	$-2/5$	$2/5$	0	1	0	48

Pivot on the $1/5$ in the p_2 column:

x_1	x_2	x_3	x_4	s_1	p_2	a_2	a_3	s_4	$-\xi$	RHS
0	0	0	0	0	0	1	1	0	1	0
-1	2	0	-3	1	0	0	1	0	0	96
-4	-1	1	-1	0	0	0	1	0	0	6
-21	-5	0	-5	0	1	-1	5	0	0	10
-8	-1	0	-1	0	0	0	2	1	0	52

All entries in the artificial objective row are nonnegative, so Phase I is complete. The artificial objective value is 0, so Phase I was successful. All artificial variables are nonbasic, so we have a basic feasible solution, which is $x_1 = 0$, $x_2 = 0$, $x_3 = 6$, $x_4 = 0$.

Another approach is to ask Maple to optimize the constant function 0 subject to the given constraints.

```
> restart;
> with(Optimization);
```

$[\mathit{ImportMPS}, \mathit{Interactive}, \mathit{LPSolve}, \mathit{LSSolve}, \mathit{Maximize}, \mathit{Minimize}, \mathit{NLPSolve}, \mathit{QPSolve}]$

```
> constraints:=[3*x1+3*x2-x3-2*x4<=90,x1+5*x3>=20,-4*x1-x2+x3-x4=6,
    x2-2*x3+x4<=40];
```

$\mathit{constraints} := [3\,x1 + 3\,x2 - x3 - 2\,x4 \le 90, 20 \le x1 + 5\,x3, -4\,x1 - x2 + x3 - x4 = 6, x2 - 2x3 + x4 \le 40]$

```
> LPSolve(0,constraints,assume=nonnegative);
```

$$[0., [x1 = 0., x2 = 0., x3 = 6., x4 = 0.]]$$

So Maple also produces the basic feasible solution $x_1 = 0$, $x_2 = 0$, $x_3 = 6$, $x_4 = 0$.

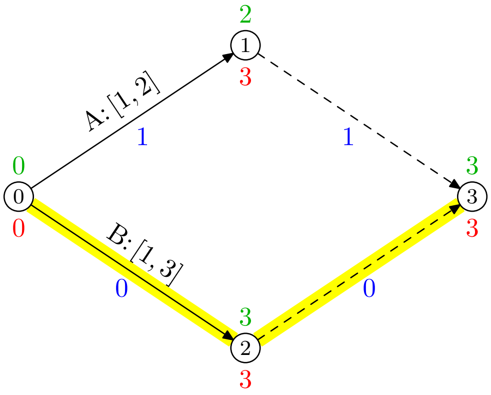
Look at the x_2 column in the tableau at the end of Phase I: it is a nonbasic column with a zero in the artificial objective row and a positive entry below. This means that we can pivot on that column to get another basic feasible solution. If we do this, we find the other extreme point of the feasible region, which is $x_1 = 0$, $x_2 = 48$, $x_3 = 54$, $x_4 = 0$. □

4. Consider a project consisting of several activities, each having a usual time and a set of immediate prerequisites. Some of the activities can be sped up; these activities additionally have a crash time and a per-unit speedup cost. (This is the kind of problem we considered when we discussed the critical path method.) Suppose that the overall project needs to be sped up to meet a deadline, and it is desired to do so at minimum total speedup cost. Is it true that only the critical activities (under the usual times) are candidates to be sped up? If so, explain why. If not, give a counterexample.

▷ **Solution.** No, activities that are not critical under the usual times may also be candidates to be sped up. For example:

Activity	Immediate prerequisites	Usual time (days)	Crash time (days)
A	—	2	1
B	—	3	1

The CPM network for this project is shown below, along with values calculated from the usual times: earliest times (written in green above the nodes), latest times (written in red below the nodes), and floats (written in blue below the arcs). The critical path under the usual times is highlighted in yellow.



In this project, only activity B is critical under the usual times. But if the project is to be sped up to meet a deadline of 1 day, then both A and B must be sped up. □

5. An important communications line is to be built between locations s and t in a dangerous, disaster-prone area. The line cannot be built directly; it will need to be built as a sequence of links joining intermediate points a, b, . . . , g. The following table shows the estimated probabilities of *failure* of a link built between pairs of points. (Not all pairs of points are possible locations for a communications link; impossible locations are indicated with a dash.) If any link fails, then the entire communications line fails. Determine a route for the line that minimizes the probability of failure.

	s	a	b	c	d	e	f	g	t
s	—	1.5%	—	—	—	—	3.0%	4.0%	—
a	1.5%	—	—	—	5.0%	—	2.0%	—	—
b	—	—	—	6.0%	8.0%	—	—	—	0.5%
c	—	—	6.0%	—	—	2.0%	3.5%	7.0%	—
d	—	5.0%	8.0%	—	—	—	2.5%	—	—
e	—	—	—	2.0%	—	—	—	5.5%	5.0%
f	3.0%	2.0%	—	3.5%	2.5%	—	—	1.0%	—
g	4.0%	—	—	7.0%	—	5.5%	1.0%	—	—
t	—	—	0.5%	—	—	5.0%	—	—	—

▷ **Solution.** If the communications line is built using k links, and the probabilities of failure of those links are p_1, p_2, …, p_k, respectively, then the probability that the line does *not* fail is

$$\prod_{i=1}^{k}(1-p_i),$$

assuming that the events of link failure are mutually independent. We aim to maximize this quantity, which we can do by maximizing its *logarithm*:

$$\log\left[\prod_{i=1}^{k}(1-p_i)\right] = \sum_{i=1}^{k}\log(1-p_i).$$

Since $0 < p_i < 1$ for all i, the value of $\log(1-p_i)$ will be negative, so equivalently we aim to *minimize* the positive sum

$$\sum_{i=1}^{k}[-\log(1-p_i)].$$

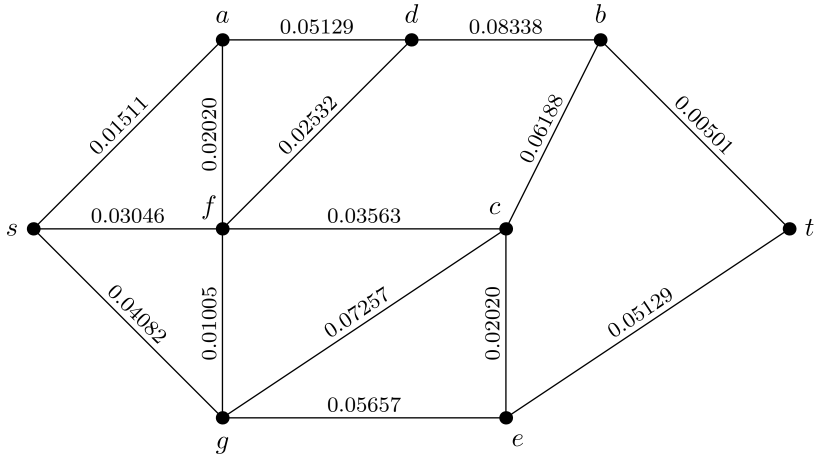
Therefore, what we are seeking is a minimum-cost path (i.e., a "shortest" path) from s to t, where the cost of each arc (i,j) is $-\log(1-p_{ij})$.

So we modify the cost table, replacing each probability p with (a rational approximation of) $-\log(1-p)$:

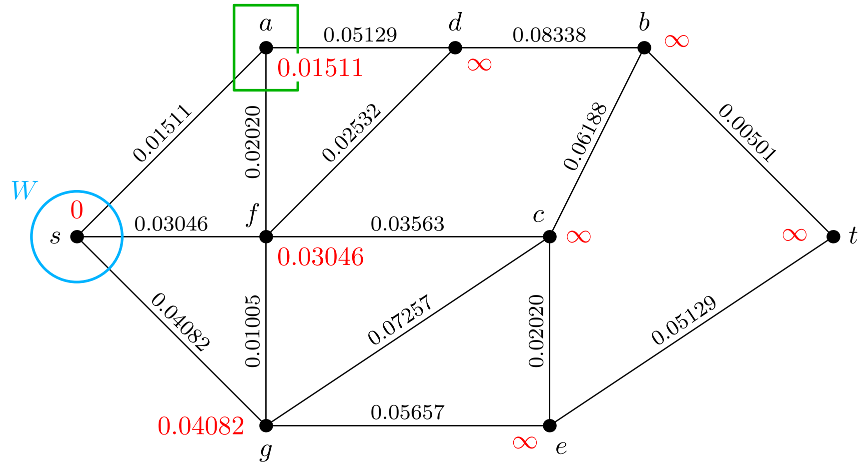
	s	a	b	c	d	e	f	g	t
s	—	0.01511	—	—	—	—	0.03046	0.04082	—
a	0.01511	—	—	—	0.05129	—	0.02020	—	—
b	—	—	—	0.06188	0.08338	—	—	—	0.00501
c	—	—	0.06188	—	—	0.02020	0.03563	0.07257	—
d	—	0.05129	0.08338	—	—	—	0.02532	—	—
e	—	—	—	0.02020	—	—	—	0.05657	0.05129
f	0.03046	0.02020	—	0.03563	0.02532	—	—	0.01005	—
g	0.04082	—	—	0.07257	—	0.05657	0.01005	—	—
t	—	—	0.00501	—	—	0.05129	—	—	—

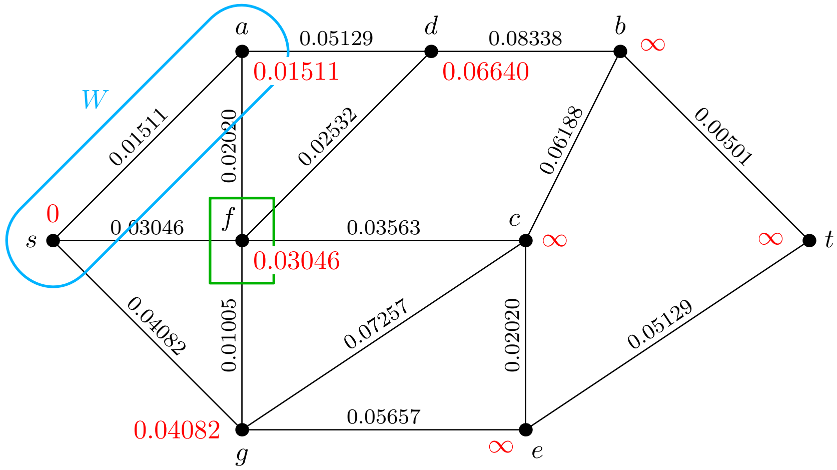
Note that because these probabilities are rather close to zero, these costs are not too far from the probabilities themselves. This is a consequence of the fact that $\log(1+x) \approx x$ for $x \approx 0$, which comes from the Maclaurin series for $\log(1+x)$. So it's not totally invalid to use the probabilities themselves as the costs.

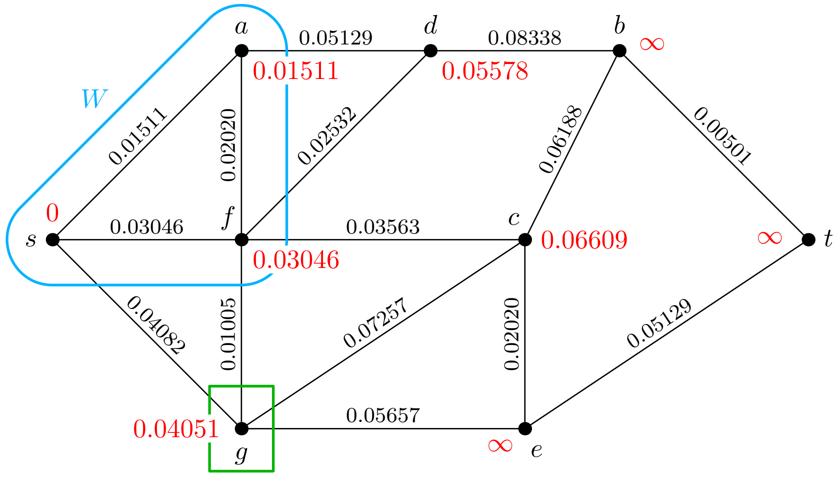
Our goal, then, is to find a shortest path from s to t in the following graph.

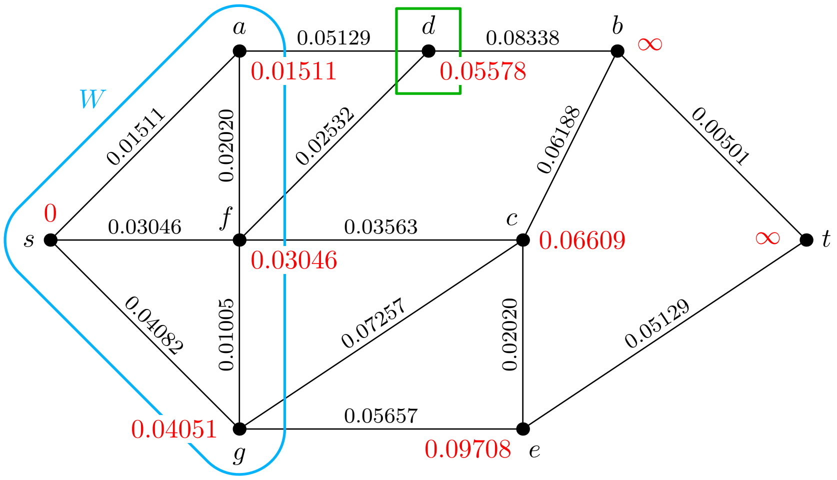


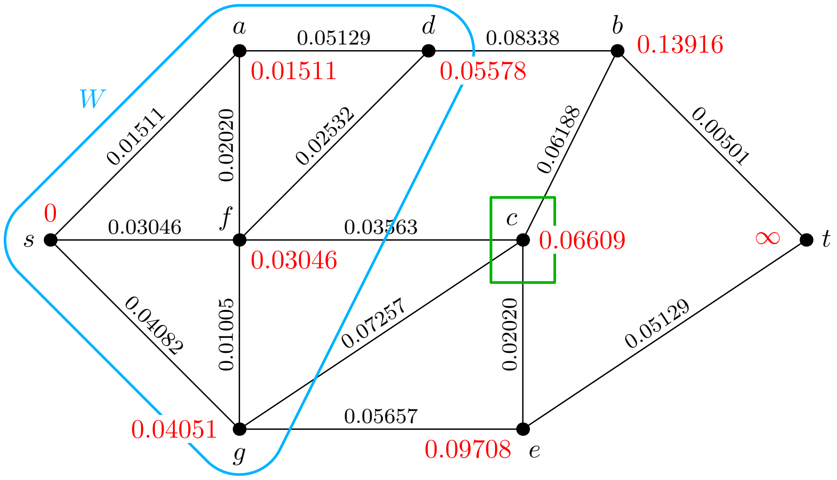
One approach is to use Dijkstra's algorithm. The iterations of Dijkstra's algorithm are shown in the figures below. In each figure, the edge weights are circled, the vertex labels (denoting the tentative distance from s) are shown in red, the current set W of vertices whose distances from s have definitely been determined is outlined in light blue, and the vertex x outside W with the smallest vertex label is boxed in green.

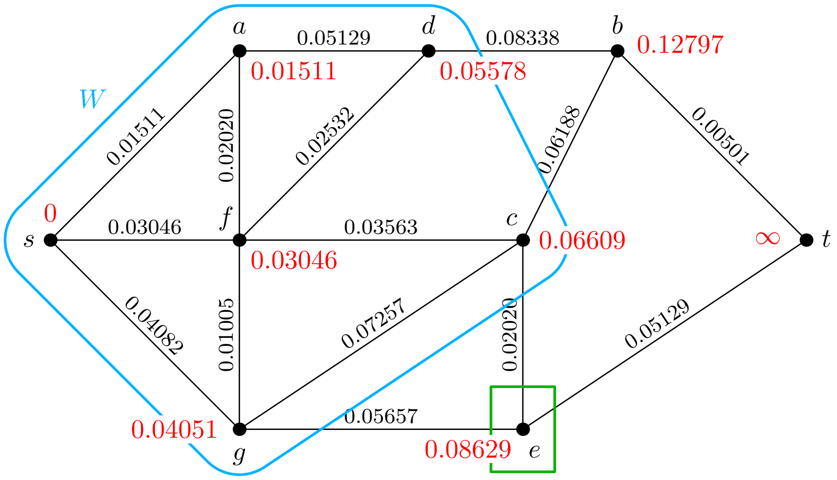


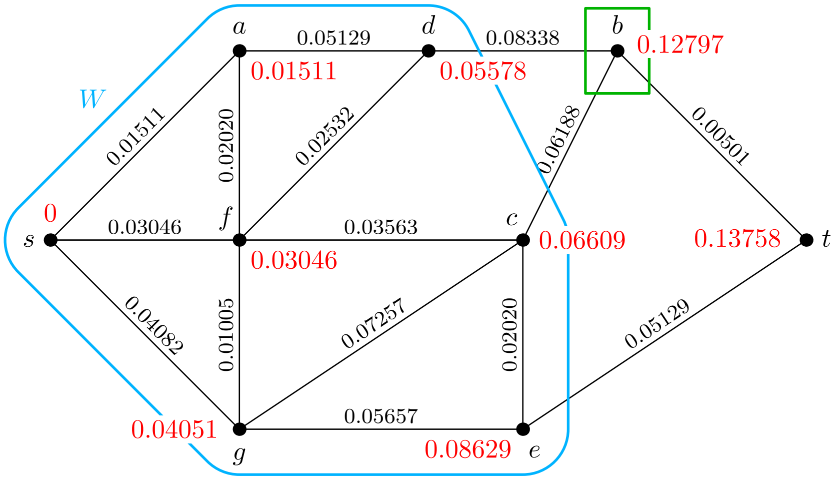


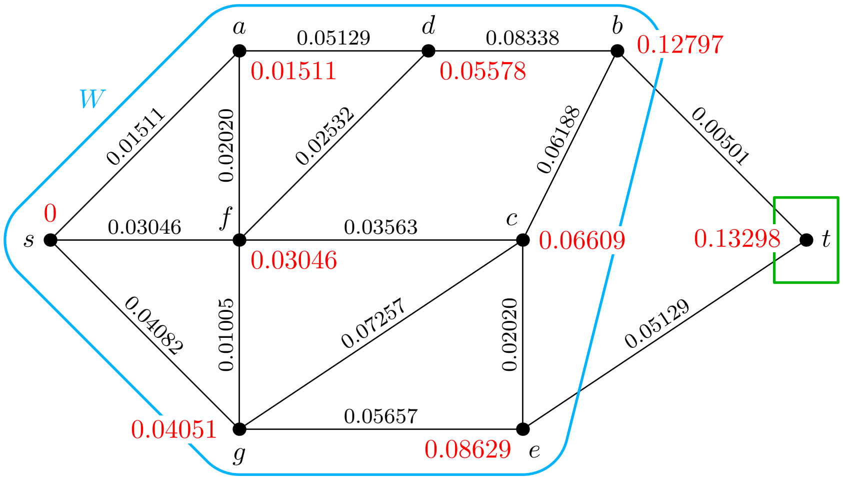


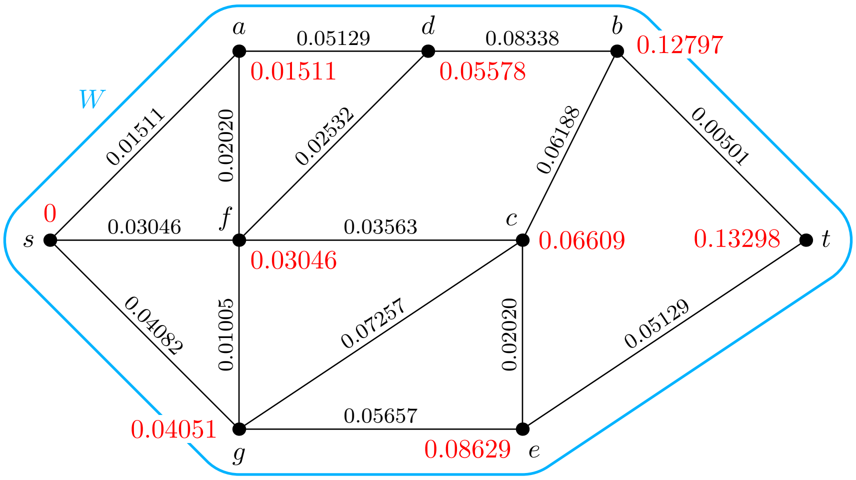

W
a
d
b
∞
0.05129
0.08338
0.01511
0.05578
0.01511
0.02020
0.02532
0.06188
0.00501
0
s
0.03046
f
0.03563
c
0.06609
∞
t
0.03046
0.04082
0.01005
0.07257
0.02020
0.05129
0.04051
g
0.05657
0.09708
e


W
a
d
b
0.13916
0.05129
0.08338
0.01511
0.05578
0.01511
0.02020
0.02532
0.06188
0.00501
0
s
0.03046
f
0.03563
c
0.06609
∞
t
0.03046
0.04082
0.01005
0.07257
0.02020
0.05129
0.04051
g
0.05657
0.09708
e

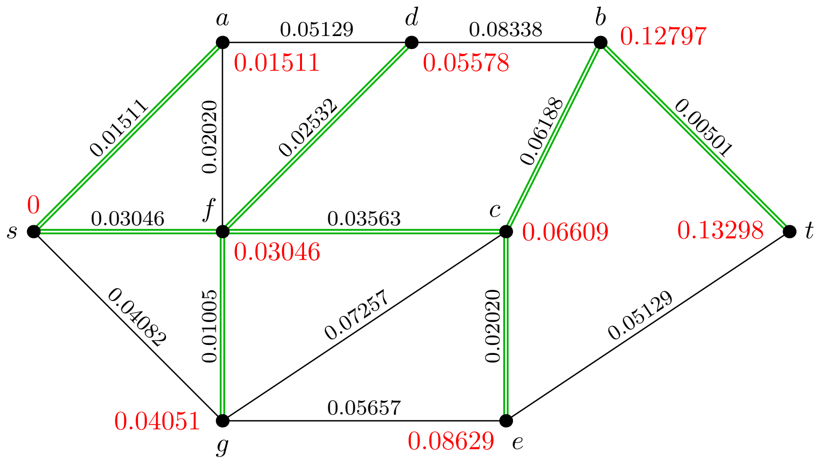

W
a
d
b
0.12797
0.05129
0.08338
0.01511
0.05578
0.01511
0.02020
0.02532
0.06188
0.00501
0
s
0.03046
f
0.03563
c
0.06609
∞
t
0.03046
0.04082
0.01005
0.07257
0.02020
0.05129
0.04051
g
0.05657
0.08629
e







Now all vertices are included in W, so the labels give the shortest distance from s for each vertex. To find a shortest path from s to any vertex, we identify the admissible edges, which are those edges whose weight equals the difference of the vertex labels at their endpoints. The admissible edges are shown as double green lines in the figure below.



From this figure, we see that the shortest path from s to t is s–f–c–b–t. The cost of this path in the graph is 0.13298, which is the negative of the logarithm of the probability that the corresponding communications line does *not* fail; so the probability of failure of this line is $1 - e^{-0.13298} \approx 12.45\%$.

Another approach is to formulate a linear program to find the shortest path from s to t in the graph. The node-arc LP formulation is shown below. For each edge (i, j) in the graph, there are two variables f_{ij} and f_{ji} denoting the flow from i to j and from j to i, respectively. (Recall that the node-arc LP formulation for the shortest path is based on a *directed* graph. The graph here is undirected, so flow can pass through an edge in either direction.) The flow-balance constraints enforce the conditions that the net outflow at every node must be 0, except at node s where it is 1 and at node t where it is -1. The objective function is the cost of the flow.

$$\begin{array}{lrl}
\text{minimize} & \displaystyle\sum_{\text{edges }(i,j)} [-\log(1-p_{ij})](f_{ij}+f_{ji}) & \\
\text{subject to} & f_{sa}+f_{sf}+f_{sg}-f_{as}-f_{fs}-f_{gs} = & 1 \\
& f_{as}+f_{ad}+f_{af}-f_{sa}-f_{da}-f_{fa} = & 0 \\
& f_{bc}+f_{bd}+f_{bt}-f_{cb}-f_{db}-f_{tb} = & 0 \\
& f_{cb}+f_{ce}+f_{cf}+f_{cg}-f_{bc}-f_{ec}-f_{fc}-f_{gc} = & 0 \\
& f_{da}+f_{db}+f_{df}-f_{ad}-f_{bd}-f_{ad} = & 0 \\
& f_{ec}+f_{eg}+f_{et}-f_{ce}-f_{ge}-f_{te} = & 0 \\
& f_{fs}+f_{fa}+f_{fc}+f_{fd}+f_{fg}-f_{sf}-f_{af}-f_{cf}-f_{df}-f_{gf} = & 0 \\
& f_{gs}+f_{gc}+f_{ge}+f_{gf}-f_{sg}-f_{cg}-f_{eg}-f_{fg} = & 0 \\
& f_{tb}+f_{te}-f_{bt}-f_{et} = & -1 \\
& \text{all variables nonnegative.} &
\end{array}$$

The following Maple worksheet solves this linear program.

```
> restart;
> with(Optimization);
```

[*ImportMPS*, *Interactive*, *LPSolve*, *LSSolve*, *Maximize*, *Minimize*, *NLPSolve*,
QPSolve]

```
> cost:=(sa,sf,sg,as,ad,af,bc,bd,bt,cb,ce,cf,cg,da,db,df,ec,eg,et,
    fs,fa,fc,fd,fg,gs,gc,ge,gf,tb,te)->-log(1-0.015)*(sa+as)
    -log(1-0.03)*(sf+fs)-log(1-0.04)*(sg+gs)-log(1-0.05)*(ad+da)
    -log(1-0.02)*(af+fa)-log(1-0.06)*(bc+cb)-log(1-0.08)*(bd+db)
    -log(1-0.005)*(bt+tb)-log(1-0.02)*(ce+ec)-log(1-0.035)*(cf+fc)
    -log(1-0.07)*(cg+gc)-log(1-0.025)*(df+fd)-log(1-0.055)*(eg+ge)
    -log(1-0.05)*(et+te)-log(1-0.01)*(fg+gf);
```

$$\begin{aligned}
\mathit{cost} := {} & (\mathit{sa}, \mathit{sf}, \mathit{sg}, \mathit{as}, \mathit{ad}, \mathit{af}, \mathit{bc}, \mathit{bd}, \mathit{bt}, \mathit{cb}, \mathit{ce}, \mathit{cf}, \mathit{cg}, \mathit{da}, \mathit{db}, \mathit{df}, \mathit{ec}, \mathit{eg}, \mathit{et}, \mathit{fs}, \mathit{fa}, \mathit{fc},\\
& \mathit{fd}, \mathit{fg}, \mathit{gs}, \mathit{gc}, \mathit{ge}, \mathit{gf}, \mathit{tb}, \mathit{te}) \to -\log(0.985)\,(\mathit{sa} + \mathit{as}) - \log(0.97)\,(\mathit{sf} + \mathit{fs})\\
& - \log(0.96)\,(\mathit{sg}+\mathit{gs}) - \log(0.95)\,(\mathit{ad}+\mathit{da}) - \log(0.98)\,(\mathit{af}+\mathit{fa}) - \log(0.94)\,(\mathit{bc}+\mathit{cb})\\
& - \log(0.92)\,(\mathit{bd} + \mathit{db}) - \log(0.995)\,(\mathit{bt} + \mathit{tb}) - \log(0.98)\,(\mathit{ce} + \mathit{ec})\\
& - \log(0.965)\,(\mathit{cf} + \mathit{fc}) - \log(0.93)\,(\mathit{cg} + \mathit{gc}) - \log(0.975)\,(\mathit{df} + \mathit{fd})\\
& - \log(0.945)\,(\mathit{eg} + \mathit{ge}) - \log(0.95)\,(\mathit{et} + \mathit{te}) - \log(0.99)\,(\mathit{fg} + \mathit{gf})
\end{aligned}$$

```
> flow:=[sa+sf+sg-as-fs-gs=1,as+ad+af-sa-da-fa=0,
    bc+bd+bt-cb-db-tb=0,cb+ce+cf+cg-bc-ec-fc-gc=0,
    da+db+df-ad-bd-fd=0,ec+eg+et-ce-ge-te=0,
    fs+fa+fc+fd+fg-sf-af-cf-df-gf=0,gs+gc+ge+gf-sg-cg-eg-fg=0,
    tb+te-bt-et=-1];
```

$$\begin{aligned}
\mathit{flow} := {} & [\mathit{sa} + \mathit{sf} + \mathit{sg} - \mathit{as} - \mathit{fs} - \mathit{gs} = 1, \mathit{as} + \mathit{ad} + \mathit{af} - \mathit{sa} - \mathit{da} - \mathit{fa} = 0,\\
& \mathit{bc} + \mathit{bd} + \mathit{bt} - \mathit{cb} - \mathit{db} - \mathit{tb} = 0, \mathit{cb} + \mathit{ce} + \mathit{cf} + \mathit{cg} - \mathit{bc} - \mathit{ec} - \mathit{fc} - \mathit{gc} = 0,\\
& \mathit{da} + \mathit{db} + \mathit{df} - \mathit{ad} - \mathit{bd} - \mathit{fd} = 0, \mathit{ec} + \mathit{eg} + \mathit{et} - \mathit{ce} - \mathit{ge} - \mathit{te} = 0,\\
& \mathit{fs} + \mathit{fa} + \mathit{fc} + \mathit{fd} + \mathit{fg} - \mathit{sf} - \mathit{af} - \mathit{cf} - \mathit{df} - \mathit{gf} = 0,\\
& \mathit{gs} + \mathit{gc} + \mathit{ge} + \mathit{gf} - \mathit{sg} - \mathit{cg} - \mathit{eg} - \mathit{fg} = 0, \mathit{tb} + \mathit{te} - \mathit{bt} - \mathit{et} = -1]
\end{aligned}$$

```
> LPSolve(cost(sa,sf,sg,as,ad,af,bc,bd,bt,cb,ce,cf,cg,da,db,df,ec,
    eg,et,fs,fa,fc,fd,fg,gs,gc,ge,gf,tb,te),flow,
    assume=nonnegative);
```

$$\begin{aligned}
\big[& 0.132974330420616, [\mathit{ad} = 0., \mathit{af} = 1.37679669262847\ 10^{-9}, \mathit{as} = 0., \mathit{bc} = 0.,\\
& \mathit{bd} = 0., \mathit{bt} = 0.999999998623202, \mathit{cb} = 0.999999995525409, \mathit{ce} = 0., \mathit{cf} = 0.,\\
& \mathit{cg} = 0., \mathit{da} = 0., \mathit{db} = 0., \mathit{df} = 0., \mathit{ec} = 0., \mathit{eg} = 0.,\\
& \mathit{et} = 3.44199262913333\ 10^{-10}, \mathit{fa} = 0., \mathit{fc} = 0.999999997934804,\\
& \mathit{fd} = 2.40939493661442\ 10^{-9}, \mathit{fg} = 0., \mathit{fs} = 0., \mathit{gc} = 0.,\\
& \mathit{ge} = 6.88398525826666\ 10^{-10}, \mathit{gf} = 0., \mathit{gs} = 0., \mathit{sa} = 0.,\\
& \mathit{sf} = 0.999999998967403, \mathit{sg} = 0., \mathit{tb} = 0., \mathit{te} = 0.]\big]
\end{aligned}$$

The optimal values of all variables are either 0 or 1 (although floating-point roundoff error is evident in the Maple output), and the shortest path is indicated by the variables that have the value 1 in the optimal solution: f_{sf}, f_{fc}, f_{cb}, and f_{bt}, which correspond to the route *s*–*f*–*c*–*b*–*t*. The optimal objective value reported by Maple agrees with the value we determined via Dijkstra's algorithm. □