VASCULAR BRANCHING

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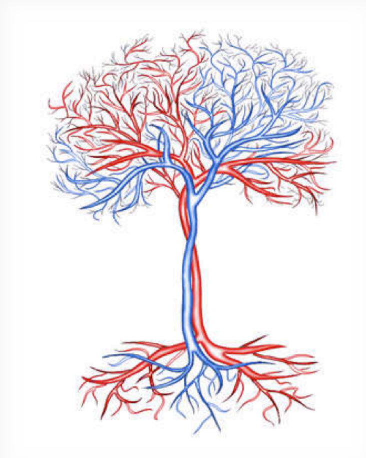
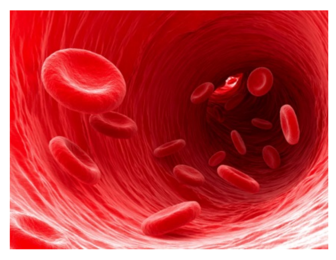


The math to the biology

VASCULAR SYSTEM

- The blood vascular system consists of blood vessels (arteries, arterioles, capillaries and veins) that convey blood around the body. The pressure difference between the ends of the vessel supplies the driving force for the flow of blood.
- This system should work so as to **minimize the energy** expended by the heart in pumping the blood. This energy is reduced when the **resistance of blood is lowered.**
- Poiseulle set up an equation that tells us about the factors resistance depends on:
- R-Resistance
- C-Viscosity
- L-Length
- R-Radius of Vessel

$$R = C\frac{L}{r^4}$$



- Understanding the equation:

1.) **As radius r decreases, blood pressure and resistance increases**
It's hard to push a lot of blood through a thin tube!
With smaller radius, the same amount of blood has to flow through the smaller volume, and since blood in incompressible, it exerts more pressure and blood flows less easily.

2.) **More viscosity C , more resistance:**
Viscosity is a type of internal friction between different layers of fluid. In blood, the viscosity is due to the cohesive forces between the molecules making it difficult to flow easily.

OBJECTIVE FUNCTION

- The aim of the function is to reduce the resistance of blood flow in **branching vessels**.
- By finding the trigonometric values of this case:

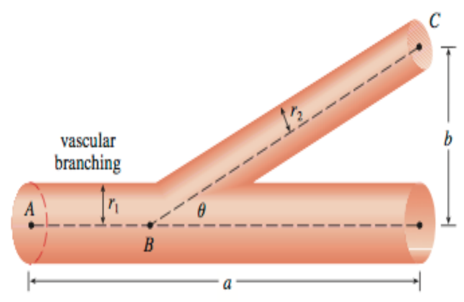
$$cot\theta = \frac{K}{b}$$

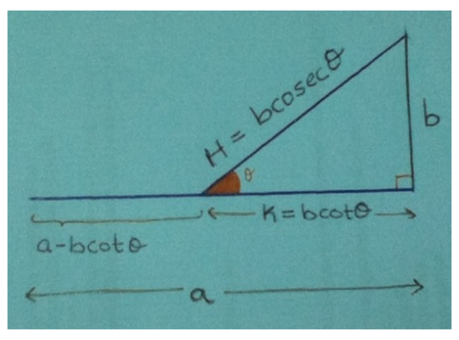
$$\boldsymbol{bcot\theta = K}$$

$$\sin\theta = \frac{b}{H}$$

$$H = \frac{b}{sin\theta}$$

$$\boldsymbol{H = bcosec\theta}$$





We know from Poiseulle's theorem,

$$R = C\frac{L}{r^4}$$

In this case:

$$L_1 = a - bcot\theta$$
$$r = r_1$$
$$L_2 = bcosec\theta$$
$$r = r_2$$

Substituting these values,

$$\boldsymbol{R = C\left[\frac{a - bcot\theta}{r_1^4} - \frac{bcosec\theta}{r_2^4}\right]}$$

DOMAIN

- At $\theta = 0°, 180°, 360°$:
$sin\theta = 0$ and so the function is undefined

$$cot\theta = \frac{cos\theta}{sin\theta} \qquad cosec\theta = \frac{1}{sin\theta}$$

- Practically, it would be silly for a vessel to not branch.
- Mathematically, the function is defined at all values of $\theta\ except\ 0°, 180°, 360°$.
- Practically, the function is defined at $0 < \theta < 90$ because there does not exist many vessels in our body that branch at an angle greater than 180° because the sine of that angle is a –ve value.
- **Domain: $0 < \theta < 90$**

FIRST DIFFERENTIAL

- On differentiating our objective function we get:

$$\frac{dR}{d\theta} = cb\left[\frac{cosec^2\theta}{r_1^4} - \frac{cosec\theta cot\theta}{r_2^4}\right]$$

CRITICAL NUMBERS

- To obtain the critical number, we equate the first differential to 0 and find the optimum angle θ in terms of r_1 and r_2. $(\frac{dR}{d\theta} = 0)$

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

$$\boldsymbol{\theta = cos^{-1}\frac{r_2^4}{r_1^4}}$$

- The critical number obtained is a positive value which means that the function is minimum at this point.

SECOND DIFFERENTIAL

- We do this step in order to verify that the critical number we have is a minimum when applied to our objective function.

$$\boldsymbol{\frac{d^2R}{d\theta^2} = cb\left[\frac{cosec\theta(cot^2\theta + cosec^2\theta)}{r_2^4} - \frac{2cosec^2\theta cot\theta}{r_1^4}\right]}$$

- When we plug in $\theta = \cos^{-1}\frac{r_2^4}{r_1^4}$ and solve, we get.

$$\boldsymbol{\frac{d^2R}{d\theta^2} = cb\left[\frac{r_1^4}{(r_1^8 - r_2^8)^{(\frac{3}{2})}}\right]\left[\frac{r_2^8 + r_1^8}{r_2^4} - \frac{2r_2^8 r_1^4}{r_1^4}\right]}$$

Since $r_1 > r_2$, we assumed a permissible value and tested that in the second derivative. Since it is a positive value , our critical number is the point of minima.

Example: At $r_1 = 1$ & $r_2 = 0.6$:

$\frac{d^2R}{d\theta^2} = 64cb > 0$ → (Since it is +ve, it is a point of minimum) (Note : C & b are always +ve)

θ VALUE at $r_2 = \frac{2}{3}r_1$

- Now that we have found the angle at which the resistance is minimum, we are presented with a case where, $\boldsymbol{r_2 = \frac{2}{3}r_1}$
- Then the optimum branching angle is :

$$\theta = \cos^{-1}\frac{r_2^4}{r_1^4}$$

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

$$\cos\theta = \frac{\left(\frac{2}{3}r_1\right)^4}{r_1^4} = \left(\frac{2^4}{3^4}\right)$$

$$\cos\theta = \frac{16}{81}$$

$$\theta = \cos^{-1}0.19$$

$$\boldsymbol{\theta = 78.63°}$$

CONCLUSION

- Resistance is minimum when angle between the vessels is:

$$\theta = \cos^{-1}\frac{r_2^4}{r_1^4} \qquad where\ r_1 > r_2$$

- This problem shows that the fundamentals of calculus can be applied even to the field of biology.

- A tiny change in angle can make a huge difference in conserving energy in our body. The minimum angle we found can be used in the field of making artificial organs and research work.

- Mathematics is hence, a subject which plays a very important role in every aspect of life.