SATELLITE ORBITS

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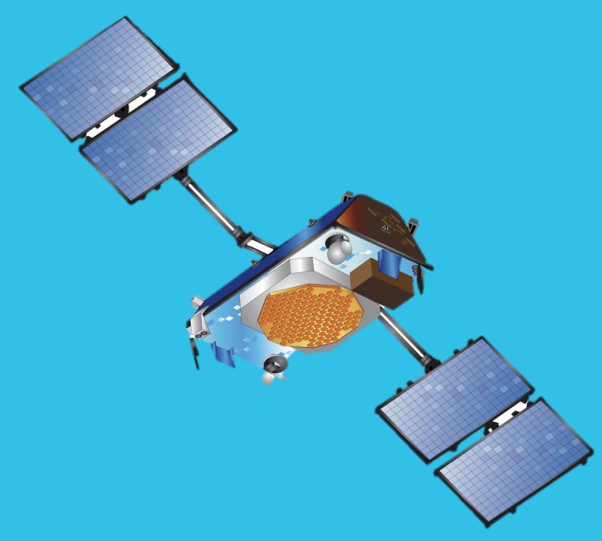


Introduction

The elliptical trajectory of a **satellite** is modelled by the equation:

$$\frac{x^2}{a^2} - \frac{x}{a} + \frac{y^2}{b^2} = 1$$

where x and y represent the x-position and y-position of the satellite relative to the Earth respectively. a and b are constants related to the lengths of the major and minor axes of the ellipse. Our aim is to figure out the **maximum** and **minimum distances** of the satellite from the Earth.

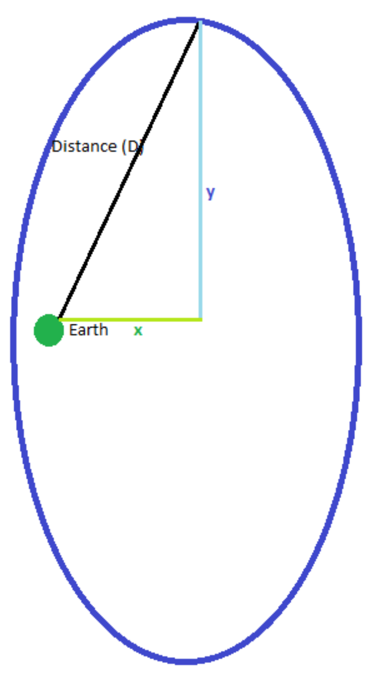
Objective Function

According to **Pythagoras' theorem**
D(x,y) = $\sqrt{(x^2+y^2)}$, where D is distance.
To reduce this **constraint** to one variable, we **solve** the original equation for y. We get:

y = $\sqrt{b^2 + b^2 \times \frac{x}{a} - b^2 \times \frac{x^2}{a^2}}$

Inputting this into the **objective function:**

D(x) = $\sqrt{\left(1 - \frac{b^2}{a^2}\right)x^2 + b^2 \times \frac{x}{a} + b^2}$



Domain of Objective Function

The equation of the ellipse we have is:

$$\frac{x^2}{a^2} - \frac{x}{a} + \frac{y^2}{b^2} = 1$$

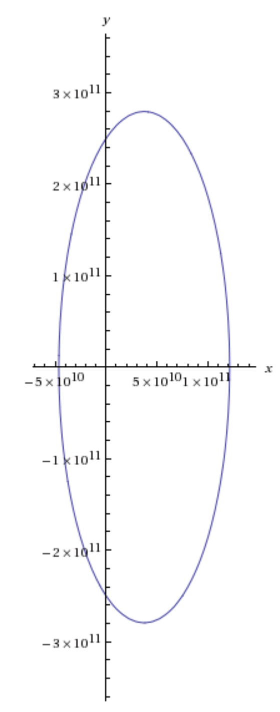
The **standard form** for ellipses is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

By completing the square in x and y, we get the standard form of the equation as:

$$\frac{\left(x - \frac{a}{2}\right)^2}{\left(\sqrt{5} \times \frac{a}{2}\right)^2} + \frac{y^2}{\left(\sqrt{5} \times \frac{b}{2}\right)^2} = 1$$

So, the **domain** is [h-a, h+a] = [$\frac{a}{2} - \frac{\sqrt{5}\,a}{2}$, $\frac{a}{2} + \sqrt{5} \times \frac{a}{2}$]



Derivative of objective function

Using the **chain rule**, **power rule** and **constant multiple rule**, we get that:

$$\frac{dD}{dx} = D'(x) = \frac{2\left(1 - \frac{b^2}{a^2}\right)x + \frac{b^2}{a^2}}{2\sqrt{\left(1 - \frac{b^2}{a^2}\right)x^2 + b^2 \times \frac{x}{a^2} + b^2}}$$

To find critical numbers, we will set the derivative equal to 0 and solve, and also find values of x for which the derivative does not exist.

Critical Numbers

The derivative is equal to 0

D'(x) = 0, so the numerator = 0

Solving for x, we get x = $\frac{b^2}{b^2 - a^2}$

The derivative is not defined

D'(x) is undefined, so the denominator = 0

Simplifying that, we get

$$\left(1 - \frac{b^2}{a^2}\right)x^2 + b^2 \times \frac{x}{a^2} + b^2 = 0$$

Which is basically saying D(x) = 0 which is impossible because then the satellite would crash into Earth.

So the only critical number is x = $\frac{b^2}{b^2 - a^2}$

Finding the max and min

The original problem states a = 7.5 x 10^{10}, b = 2.5 x 10^{10}

So, we evaluate D(x) at the following points:

Point	Approximate Value of x	Approximate value of D(x)
Lower Endpoint	-4.6353 x 10^{10}	**4.6353 x 10^{10}**
Upper Endpoint	12.1353 x 10^{10}	**12.1353 x 10^{10}**
$\frac{b^2}{b^2 - a^2}$	1.0989 x 10^{10}	2.654 x 10^{11}

So, the minimum value of the function is 4.6353 x 10^{10} and the maximum value of the function is 2.654 x 10^{11}.

Answer to the question

Using the process of **differentiation** and **finding critical values**, we were able to locate the **maximum** and **minimum** values of the distance function.

Our final answer is:

In terms of a and b, the minimum occurs at $\frac{a}{2} - \frac{\sqrt{5}\,a}{2}$ and the maximum occurs at $\frac{b^2}{b^2 - a^2}$

The **maximum** distance from the Earth is **2.654x10^{11}** m

The **minimum** distance from the Earth is **4.6353x10^{10}** m

This process is an example of the uses of differential calculus in real life applications