Another interpretation of duality

(Based on an example in Chapter 5 of *Linear Programming: Foundations and Extensions*, third edition, by Robert J. Vanderbei.)

Consider the following maximization problem:

$$\begin{array}{lrl} \text{maximize} & 8x_1 + 5x_2 + 6x_3 & \\ \text{subject to} & x_1 + \ x_2 + \ x_3 \le 8 & \quad (1) \\ & 4x_1 + 2x_2 - \ x_3 \le 7 & \quad (2) \\ & x_1,\, x_2,\, x_3 \ge 0. & \end{array}$$

Let z^* denote the optimal value of the objective function in this maximization problem. We would like to determine upper and lower bounds for z^*.

Finding lower bounds is easy: every feasible solution demonstrates a lower bound for z^*. For example, the feasible solution $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ yields an objective value of $8(1)+5(2)+6(3) = 36$, so we know that $z^* \ge 36$. If we choose a different feasible solution, we can get another lower bound for z^*. For instance, the feasible solution $x_1 = 0$, $x_2 = 5$, $x_3 = 3$ yields an objective value of $8(0) + 5(5) + 6(3) = 43$, so we can now conclude that $z^* \ge 43$.

However, finding lower bounds this way is like trying to solve the maximization problem by trial and error. A better method is to use the simplex algorithm to solve the maximization problem. The simplex algorithm produces the optimal solution $x_1 = 3$, $x_2 = 0$, $x_3 = 5$, with an objective value of $8(3) + 5(0) + 6(5) = 54$. This feasible solution demonstrates that $z^* \ge 54$.

In fact we know that the simplex algorithm produces *optimal* solutions, so we know that z^* actually *equals* 54. But how could we prove this to someone who doesn't know about the simplex algorithm, who doesn't believe that the simplex algorithm always produces the optimal solution, or who doesn't want to check all the work that was done in the simplex algorithm? For this we need upper bounds for z^*.

Upper bounds for z^* come from linear combinations of the constraints in the maximization problem. For example, we can multiply constraint (*1*) by 8 and constraint (*2*) by 2 and add them together to get the following inequality:

$$\begin{array}{rl} 8(\ \ x_1 + \ \ \ x_2 + \ x_3) & \le 8(8) \\ +\,2(\ 4x_1 + \ 2x_2 - \ x_3) & \le 2(7) \\ \hline 16x_1 + 12x_2 + 6x_3 & \le 78. \end{array}$$

Since all of the variables x_1, x_2, and x_3 are nonnegative, and each of the coefficients in $16x_1+12x_2+6x_3$ is at least as large as the corresponding coefficient in the objective function $8x_1 + 5x_2 + 6x_3$, this demonstrates that $z^* \le 78$:

$$z = 8x_1 + 5x_2 + 6x_3 \le 16x_1 + 12x_2 + 6x_3 \le 78.$$

The inequality $16x_1+12x_2+6x_3 \le 78$ is a direct consequence of the constraints of the maximization problem, and we could show this inequality to a skeptic (by providing the multipliers 8 and 2) in order to prove that the objective value can be no greater than 78.

But can we do better? We can find other linear combinations of the constraints by trying other multipliers, and as long as the coefficients of the inequality we get are at least as large as the corresponding coefficients in the objective function, we will get an upper bound for z^*. Finding upper bounds this way, though, is again a trial-and-error process; we would like to be more methodical.

So let's assign variable names to the multipliers we use. We'll multiply constraint (*1*) by y_1 and multiply constraint (*2*) by y_2 to get the following inequality:

$$\begin{array}{rcccl} y_1(& x_1 + & x_2 + & x_3) & \le y_1(8) \\ +\,y_2(& 4x_1 + & 2x_2 - & x_3) & \le y_2(7) \\ \hline & (y_1 + 4y_2)x_1 + & (y_1 + 2y_2)x_2 + & (y_1 - y_2)x_3 & \le 8y_1 + 7y_2. \end{array}$$

This will give us an upper bound for z^* as long as the coefficients in this new inequality are at least as large as the corresponding coefficients in the objective function; so we need

$$\begin{aligned} y_1 + 4y_2 &\geq 8, \\ y_1 + 2y_2 &\geq 5, \\ y_1 - y_2 &\geq 6. \end{aligned}$$

We also need $y_1 \geq 0$, $y_2 \geq 0$ so that when we multiply the constraints by these values we don't flip the direction of the inequality.

If these conditions are satisfied, then we can conclude that

$$z^* \leq 8y_1 + 7y_2.$$

To make this upper bound as good as possible, we want to *minimize* the value of $8y_1 + 7y_2$. This gives us the following minimization problem:

$$\begin{array}{rr} \text{minimize} & 8y_1 + 7y_2 \\ \text{subject to} & y_1 + 4y_2 \geq 8 \\ & y_1 + 2y_2 \geq 5 \\ & y_1 - y_2 \geq 6 \\ & y_1,\, y_2 \geq 0. \end{array}$$

This minimization problem is the *dual* of the original maximization problem.

If we solve this minimization problem, we get the optimal solution $y_1 = 6.4$, $y_2 = 0.4$. Using these multipliers for the constraints of the original maximization problem, we obtain the inequality

$$\begin{array}{rl} 6.4(\ \ x_1 + \ \ \ \ x_2 + \ x_3) & \leq 6.4(8) \\ +\,0.4(\ 4x_1 + \ \ 2x_2 - \ x_3) & \leq 0.4(7) \\ \hline 8x_1 + 7.2x_2 + 6x_3 & \leq 54. \end{array}$$

This proves that the optimal value of the objective function of the maximization problem can be no greater than 54, because

$$z = 8x_1 + 5x_2 + 6x_3 \leq 8x_1 + 7.2x_2 + 6x_3 \leq 54.$$

Together with the feasible solution $x_1 = 3$, $x_2 = 0$, $x_3 = 5$ that actually gives us an objective value of 54, this provides a proof that the optimal value of the original maximization problem is indeed 54.