Monday 21 July 2014

NEWTON'S METHOD

Motivation

- Linear equations like $ax+b=0$ are easy to solve.
- Quadratic equations: $ax^2+bx+c=0$
  - Use the quadratic formula.
- Cubic equations: $ax^3+bx^2+cx+d=0$
  Quartic equations: $ax^4+bx^3+cx^2+dx+e=0$
  - Methods developed in Italy in the 1500s by Tartaglia, Cardano, Ferrari, others.
- Quintic equations: $ax^5+bx^4+cx^3+dx^2+ex+f=0$
  - Ruffini 1799, Abel 1823, Galois 1828–1832 proved that there is <u>no general algebraic method</u> for solving these equations (or polynomial equations of higher degree).
- So what do we do? <u>APPROXIMATE.</u>
- Newton's method is a technique for approximating roots of differentiable functions using calculus.

__Example.__ Solve the equation $x^5 - x + 1 = 0$.

Unfortunate fact: The solution to this equation __cannot__ be expressed exactly using addition, subtraction, multiplication, division, integer exponents, and roots.

There is __no algebraic solution__ to this equation.

But there __is__ a solution, and we can __approximate__ it.

Note that solving the equation $x^5 - x + 1 = 0$ is equivalent to finding a __root__ of the function $f(x) = x^5 - x + 1$.

Start with a guess. Try $x_0 = -1$.

Check the guess by evaluating the function:

$$f(-1) = (-1)^5 - (-1) + 1 = 1.$$

The value of the function is too large: we want $f(x) = 0$. So we will adjust our guess. How?

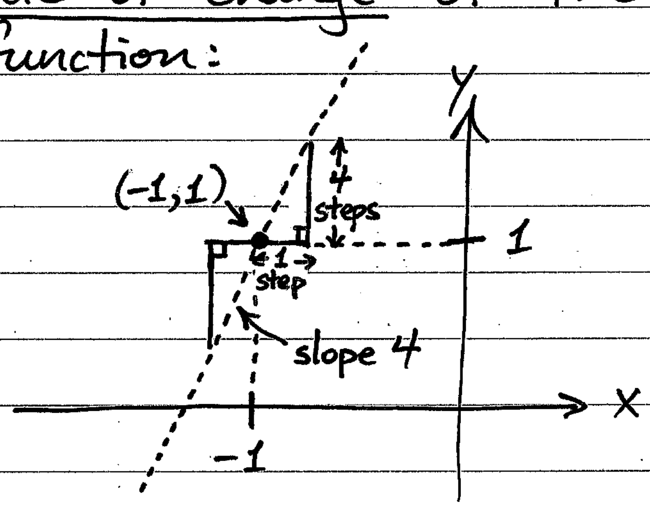
Think about the <u>derivative</u>:

$$f'(x) = 5x^4 - 1.$$

The value of the derivative at $x_0 = -1$ is

$$f'(-1) = 5(-1)^4 - 1 = 4.$$

This tells us about the <u>slope</u> of the function $f(x)$ at $x_0 = -1$, which is the <u>rate of change</u> of the value of the function:



At $x_0 = -1$, the slope of $f(x)$ is 4, which means that if we take one step to the <u>right</u> from this point, the value of $f(x)$ should <u>increase</u> by four steps, and if we take one step to the <u>left</u> the value of $f(x)$ should <u>decrease</u> by four steps.

— (Of course, $f(x)$ is not a straight line — it curves away from the tangent line. But the tangent line should be <u>similar</u> to $f(x)$ near $x_0 = -1$.)

The value of $f(x)$ was <u>too large</u> at $x_0 = -1$. We would like to <u>decrease</u> its value by 1, from 1 to 0. So we adjust our guess by moving <u>$\frac{1}{4}$ unit</u> to the left, because the slope of $f(x)$ at $x_0 = -1$ suggests that moving $\frac{1}{4}$ unit to the left will decrease the value of the function by $4(\frac{1}{4}) = 1$ unit, which is what we want.

So our new guess will be $x_1 = -\frac{5}{4}$.

Where did this new guess come from?

$$\underset{\text{new guess}}{-\frac{5}{4}} = \underset{\text{old guess}}{(-1)} - \frac{\overset{\text{value of } f(x) \text{ at old guess}}{1}}{\underset{\text{value of } f'(x) \text{ at old guess}}{4}}$$

Check the new guess:

$$f(-\tfrac{5}{4}) = (-\tfrac{5}{4})^5 - (-\tfrac{5}{4}) + 1 \approx -0.80176.$$

It's better, but we can still improve.

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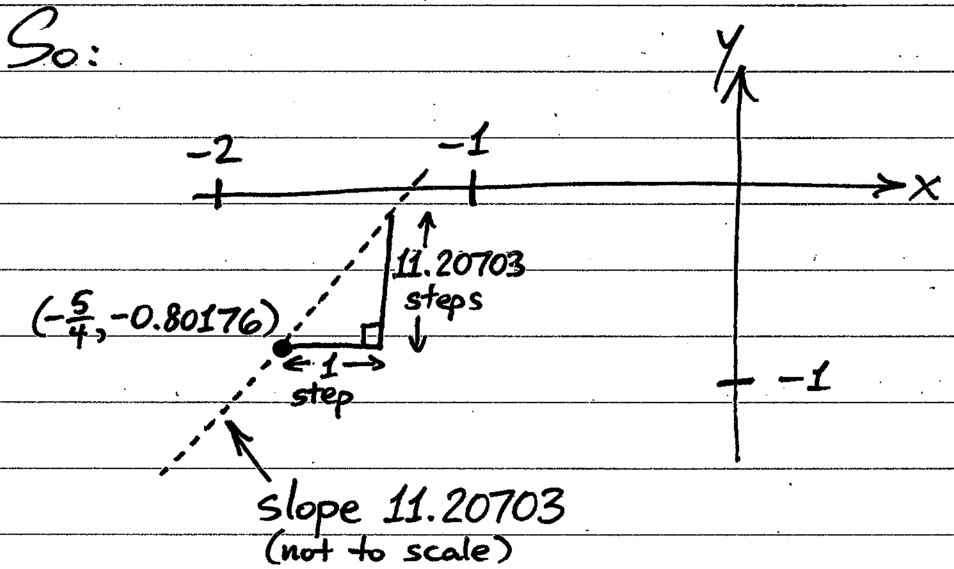
Repeat the process:

Value of the derivative at $x_1 = -\frac{5}{4}$ is

$$f'\left(-\frac{5}{4}\right) = 5\left(-\frac{5}{4}\right)^4 - 1 \approx 11.20703$$

So:



We need to increase the value of $f(x)$ by 0.80176. Every step to the right will increase the value of $f(x)$ by 11.20703 steps (we expect). So we should adjust our guess by adding $\frac{0.80176}{11.20703}$. Our next guess is

$$x_2 = -\frac{5}{4} + \frac{0.80176}{11.20703} \approx -1.17846.$$

Same as before:

two minuses make a plus

value of $f(x)$ at old guess

$$-1.17846 = \left(-\frac{5}{4}\right) - \frac{-0.80176}{11.20703}$$

new guess　　old guess　　value of $f'(x)$ at old guess

⑥

Check the new guess:

$$f(-1.17846) = (-1.17846)^5 - (-1.17846) + 1$$
$$\approx 0.09440.$$

Much closer! Let's repeat the process once more.

Value of the derivative at $x_2 = -1.17846$ is

$$f'(-1.17846) = 5(-1.17846)^4 - 1 \approx 8.64336.$$

So the next guess should be

value of $f(x)$ at old guess

$$\underbrace{x_3}_{\text{new guess}} = \underbrace{(-1.17846)}_{\text{old guess}} - \frac{0.09440}{8.64336}$$

value of $f'(x)$ at old guess

$$x_3 = -1.16754.$$

Check the new guess:

$$f(-1.16754) = (-1.16754)^5 - (-1.16754) + 1$$
$$\approx -0.00193.$$

Close enough.

— Or, if not, keep going.

Steps of Newton's Method

To approximate a root of the function $f(x)$:

1. Find the derivative, $f'(x)$.

2. Make an initial guess for the root. Call the initial guess x_0.

3. From the approximation x_i, find the next approximation x_{i+1} by using the formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

In other words, calculate the value of the function at the current approximation, $f(x_i)$, and calculate the value of the derivative at the current approximation, $f'(x_i)$. Then subtract the quotient $\frac{f(x_i)}{f'(x_i)}$ from the current approximation to get the next approximation.

4. Repeat step 3 until you get an approximation that is as accurate as you want (hopefully).

— Note: Newton's method does not always <u>converge</u>, meaning that the approximations do not always keep getting better. If it seems not to be converging, try a different initial guess. Or maybe you just have a bad function for Newton's method, or maybe the function has no roots.

⑧ **Example.** (Using a table for Newton's method.)

Approximate a solution to the equation $e^x = x^2$.

— This equation cannot be solved algebraically. There is no way to get the x out of the exponent without burying the other x inside a logarithm.

— But solving $e^x = x^2$ is equivalent to solving $e^x - x^2 = 0$, which is equivalent to finding a root of the function $f(x) = e^x - x^2$. So we can approximate a solution to the equation by using Newton's method.

$$f(x) = e^x - x^2$$

$$f'(x) = e^x - 2x$$

Initial guess: Try $x_0 = 0$. (Why not?)

Make a table:

x [guess]	$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$ [next guess]
0	1	1	−1
−1	−0.63212	2.36788	−0.73304
−0.73304	−0.05691	1.94653	−0.70381
−0.70381	−0.00065		

Close enough. Approx. solution: $\boxed{x \approx -0.70381.}$