21-110: Problem Solving in Recreational Mathematics

Homework assignment 8

Assigned Wednesday, April 14, 2010. Due Monday, April 26, 2010.

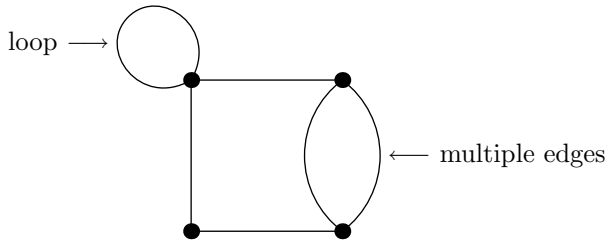
Work at least **FOUR** of the following problems, at least one of which must be from Part B. All problems are of equal weight. If you submit solutions for more than four problems, you will get credit for your best four (with the proviso that you will get credit for at most three problems from Part A).

You are welcome to work with other students, but the solutions you hand in should be written in your own words. You are not allowed to see the paper another student is going to hand in. If you do collaborate with other students, list their names. If you use other sources, cite them. Give credit where credit is due. See the syllabus for more information about academic integrity.

Hints are encrypted with a *Caesar cipher*, in which each letter is replaced by the letter three places ahead in the alphabet, wrapping around to the beginning if necessary. For example, the letter A is encrypted as D, and Y is encrypted as B. To decrypt the hints, move each letter backward three places.

— Part A —

Problem 1. A *loop* is an edge that joins a vertex to itself. *Multiple edges* are two or more edges between the same pair of vertices. (See the picture below.) A *simple graph* is a graph with no loops and no multiple edges. Draw a simple graph with at least two vertices such that no two vertices have the same degree, or explain why this is impossible.



Problem 2. Draw an example of each of the following:

(a) A graph that has an Eulerian circuit but no Hamiltonian circuit.

(b) A graph that has a Hamiltonian circuit but no Eulerian circuit.

In both cases, justify that your graph satisfies the required conditions.

Problem 3. Agnes hosts a party. Over the course of the evening some people at the party shake hands. Near the end of the party, but before any of the guests have left, Agnes gets everyone's attention and asks how many people shook hands an odd number of times. Exactly 11 guests say they have done so. Assuming that all the guests remember how many hands they shook, did Agnes shake an even number or an odd number of hands? Why?

Hint: Vhh Wkhruhp 6.5 rq sdjh 186.

Problem 4. (Problem 6.21 from *Problem Solving Through Recreational Mathematics*.) Describe a knight's tour on a 3×7 chessboard.

Hint: Vhh wkh klqw rq sdjh 402.

Problem 5. You decide that you want to visit the six largest cities in Pennsylvania (according to the 2007 population estimates from the U.S. Census Bureau). The shortest distances between these cities, according to Google Maps, are given in the distance table below.

Distances in miles	**A.**	**Erie**	**Phila.**	**Pgh.**	**R.**	**S.**
Allentown	—	371	55	279	39	75
Erie	371	—	401	128	358	297
Philadelphia	55	401	—	301	57	124
Pittsburgh	279	128	301	—	258	282
Reading	39	358	57	258	—	97
Scranton	75	297	124	282	97	—

(a) Draw a weighted graph to represent this information. (Be sure to label the vertices of the graph.)

(b) You want to start in Pittsburgh, visit all of these cities, and return to your starting point. In mathematical terms, what are you trying to find? Why?

(c) You would like to keep your total travel distance low. Use your graph from part (a) to find an efficient way to tour these six cities. Explain the method you are using and show all of your steps. What is the total distance you travel if you go this way?

— Part B —

Problem 6. (Problem 6.25 from *Problem Solving Through Recreational Mathematics.*) Three married couples want to cross a river. The only boat available is capable of holding two people at a time. This would present no difficulty were it not for the fact that the women are all very jealous, so that each woman refuses to allow her husband to be in the presence of another woman unless she herself is also present.

How should they cross the river with the least amount of rowing?

Hint: Vhh wkh klqw rq sdjh 402.

Problem 7. Eight radio stations, one in each of the eight cities in the distance table below, are to be assigned frequencies. Two stations cannot be assigned the same frequency if they are within 200 miles of each other; otherwise their signals will interfere.

Distances in miles	**Ashland**	**Bristol**	**Clinton**	**Douglas**	**Empire**	**Franklin**	**Greene**	**Huron**
Ashland	—	292	221	366	357	127	240	298
Bristol	292	—	93	195	366	329	247	419
Clinton	221	93	—	171	290	239	162	329
Douglas	366	195	171	—	223	330	171	330
Empire	357	366	290	223	—	252	136	142
Franklin	127	329	239	330	252	—	167	172
Greene	240	247	162	171	136	167	—	176
Huron	298	419	329	330	142	172	176	—

(a) Draw a conflict graph for this scenario. Explain what the vertices represent and what causes a conflict.

(b) Color the vertices of the conflict graph using as few colors as possible, according to the rules of graph coloring.

(c) Interpret your coloring from part (b). How many different frequencies are needed? How should they be assigned?

Problem 8. A map of South America requires four colors if no two countries which share a common border are to receive the same color; three colors are not enough. An easy way to see this is to consider Paraguay and its three neighbors, as shown in the map on the left below. Each of these four countries (Argentina, Bolivia, Brazil, and Paraguay) borders all of the other three. (In other words, the dual graph of this map is the complete graph on four vertices.) So four colors are necessary.

This situation does not arise in a map of the United States—there are no four states such that each of the four borders all of the other three. So, no matter which four states you choose, there will always be two of them that do not border each other. (Verify this for yourself.)

Does this mean that a map of the United States can be colored with just three colors? If so, show how it can be done by coloring the map of the 48 contiguous United States (shown on the right below, not to the same scale as the other map) using just three colors in such a way that no two bordering states receive the same color. If it cannot be done, explain, as carefully and as thoroughly as you can, why four colors are necessary.

(Note that Utah and New Mexico touch at only a single point, so they are not considered to be bordering states. The same is true for Colorado and Arizona.)

Hint: Qhydgd dqg lwv qhljkeruv.