21-110: Problem Solving in Recreational Mathematics

Homework assignment 7 solutions

Problem 1. An urn contains five red balls and three yellow balls. Two balls are drawn from the urn at random, without replacement.

(a) In this scenario, what is the experiment? What is the sample space?

(b) What is the probability that the first ball drawn is red?

(c) What is the probability that at least one of the two balls drawn is red?

(d) What is the (conditional) probability that the second ball drawn is red, given that the first ball drawn is red?

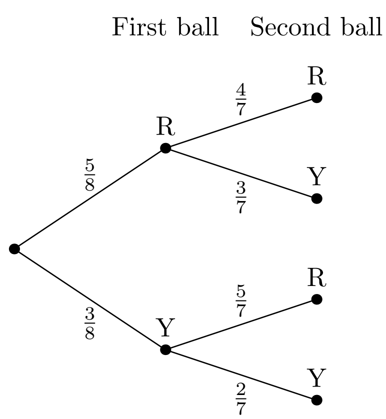
Solution.

(a) The experiment is the drawing of two balls from the urn without replacement. The sample space is the set of possible outcomes, of which there are four: drawing two red balls; drawing two yellow balls; drawing a red ball first, and then a yellow ball; and drawing a yellow ball first, and then a red ball. One way to denote the sample space is in set notation, abbreviating the colors red and yellow:

$$\text{sample space} = \{\mathrm{RR}, \mathrm{YY}, \mathrm{RY}, \mathrm{YR}\}.$$

Note that these four outcomes are *not* equally likely.

We can also represent the experiment and the possible outcomes in a probability tree diagram, as shown below. Note in particular the probabilities given for the second ball. For example, if the first ball is red, then four out of the remaining seven balls are red, so the probability that the second ball is red is 4/7 (and the probability that it is yellow is 3/7). On the other hand, if the first ball is yellow, then five out of the remaining seven balls are red, so the probability that the second ball is red is 5/7 (and the probability that it is yellow is 2/7). The probabilities of each of the four outcomes can be computed by multiplying the probabilities along the branches leading to the outcomes.



Outcome	Probability
RR	$\frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$
RY	$\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$
YR	$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$
YY	$\frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

(b) Before the first draw, five out of the eight balls in the urn are red. Each ball is equally likely to be drawn. So the probability that the first ball drawn is red is 5/8.

(c) The event that at least one of the two balls is red contains three outcomes: RR, RY, and YR. Since we know the probabilities of all of these outcomes, we can find the probability of this event by adding the probabilities of the individual outcomes. So the probability that at least one of the two balls is red is $5/14 + 15/56 + 15/56 = 25/28$.

Alternatively, we can observe that the event that at least one of the two balls is red is the complement of the event that both balls are yellow. The probability that both balls are yellow is 3/28, so the probability that it is *not* the case that both balls are yellow (i.e., the probability that at least one ball is red) is $1 - 3/28 = 25/28$.

(d) Suppose the first ball drawn is red. Then there are seven remaining balls, and four of them are red. So the conditional probability that the second ball is red, given that the first ball is red, is 4/7. □

Problem 2. Suppose you and a friend play a game. Two standard, fair, six-sided dice are thrown, and the numbers appearing on the dice are multiplied together. If this product is even, your friend gives you a quarter, but if this product is odd, you must give your friend one dollar.

(a) What is the expected value of this game for you? Round to the nearest cent.

(b) What is the expected value of this game for your friend? Round to the nearest cent.

(c) Is this game fair? Why or why not?

Solution. We begin by drawing a table of the possible dice throws. The 36 outcomes represented by the squares of the grid below are all equally likely. The shaded squares represent outcomes in which the product of the numbers on the dice is even, and the unshaded squares represent outcomes in which this product is odd.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

(a) From the table above, we see that 27 of the 36 equally likely outcomes yield an even product, so the probability that the product is even is 27/36, or 3/4. Similarly, the probability that the product is odd is 9/36, or 1/4. You will win 25¢ if the product of the numbers on the dice is even, but you will lose \$1 if the product is odd. So there is a 3/4 probability that you will gain 25¢ and a 1/4 probability that you will lose 100¢ (that is, gain −100¢). So the expected value of the game for you is

$$\tfrac{3}{4}(25¢) + \tfrac{1}{4}(-100¢) = -6.25¢.$$

Rounded to the nearest cent, this is −6¢. This means that you should expect to lose about 6¢, on average, each time you play this game.

(b) Any money that you win in this game is money that your friend loses, and any money that your friend wins is money that you lose. (Mathematicians call this a *zero-sum game.*) Therefore, if the expected value of the game for you is −6¢, the expected value of the game for your friend is 6¢. Your friend should expect to win about 6¢, on average, each time the game is played.

(c) This game is not fair, because the expected value of the game is not exactly zero. □

Problem 3. Suppose you roll a fair six-sided die six times and record the results of the rolls in order. What is the probability that you roll a "3" at least five times in a row?

Solution. There are essentially two ways in which this event can occur: either the first five rolls are all "3," or the last five rolls are all "3." Let's give names to these events; let A be the event that the first five rolls are all "3," and let B be the event that the last five rolls are all "3." The question asks for the probability that either of these events occur, which is the probability of the event $A \cup B$.

How many outcomes are there in all? In other words, what is the cardinality of the sample space? Each roll of the die has six possible outcomes, so, by the multiplication principle, in six rolls of the die there are 6^6, or 46,656, possible outcomes (because the order of the rolls is important). These outcomes are all equally likely.

How many outcomes are in the event A? The outcomes in the event A are those that look like "3, 3, 3, 3, 3, ?," where the question mark can be replaced by any number from 1 through 6. So the cardinality of A is 6.

Likewise, the outcomes in the event B are those that look like "?, 3, 3, 3, 3, 3," where the question mark can be replaced with any number from 1 through 6. So the cardinality of B is also 6.

How many outcomes are in both A and B? In other words, what is the cardinality of $A \cap B$? There is only one such outcome, namely, "3, 3, 3, 3, 3, 3." So $|A \cap B| = 1$.

By the principle of inclusion–exclusion, we have

$$|A \cup B| = |A| + |B| - |A \cap B| = 6 + 6 - 1 = 11.$$

Alternatively, we could have found $|A \cup B|$ by listing all of the elements of $A \cup B$ explicitly:

$$A \cup B = \{\, 1{,}3{,}3{,}3{,}3{,}3;\ 2{,}3{,}3{,}3{,}3{,}3;\ 3{,}3{,}3{,}3{,}3{,}1;\ 3{,}3{,}3{,}3{,}3{,}2;\ 3{,}3{,}3{,}3{,}3{,}3;\ 3{,}3{,}3{,}3{,}3{,}4;\ 3{,}3{,}3{,}3{,}3{,}5;\ 3{,}3{,}3{,}3{,}3{,}6;\ 4{,}3{,}3{,}3{,}3{,}3;\ 5{,}3{,}3{,}3{,}3{,}3;\ 6{,}3{,}3{,}3{,}3{,}3 \,\}.$$

So, using S to denote the sample space, we have

$$\Pr(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{11}{46{,}656}.$$

Therefore, the probability that a "3" is rolled at least five times in a row is 11/46,656, or about two hundredths of one percent. □

Problem 4. (From *The Colossal Book of Short Puzzles and Problems* by Martin Gardner.) A secretary types four letters to four people and addresses the four envelopes. If she inserts the letters at random, each in a different envelope, what is the probability that exactly three letters will go into the right envelopes?

Solution. If three letters go into the right envelopes, then the fourth letter must go into the fourth envelope, which must be the right one! So the event that exactly three letters go into the right envelopes contains zero outcomes, which means that the probability of this event is zero. □

Problem 5. (From *The Colossal Book of Short Puzzles and Problems* by Martin Gardner.) Bill, a student in mathematics, and his friend John, an English major, usually spun a coin on the bar to see who would pay for each round of beer. One evening Bill said: "Since I've won the last three spins, let me give you a break on the next one. You spin *two* pennies and I'll spin one. If you have more heads than I have, you win. If you don't, I win."

"Gee, thanks," said John.

On previous rounds, when one coin was spun, John's probability of winning was, of course, 1/2. What are his chances under the new arrangement?

Solution. Let's make a table of the possible outcomes. Remember, John wins if he gets *more* heads than Bill. If there is a tie, Bill wins.

Bill	John	Winner
H	H H	John
H	H T	Bill
H	T H	Bill
H	T T	Bill
T	H H	John
T	H T	John
T	T H	John
T	T T	Bill

The eight outcomes in the table above are equally likely. John wins in four of the eight cases. So the probability that John wins under the new arrangement is 4/8, or 1/2, the same as it was before. □

[Gardner points out that John's probability of winning remains 1/2 whenever he has one more coin than Bill. For example, if John has 51 coins and Bill has 50, John's probability of winning is still 1/2.]

Problem 6. Alice has three fair six-sided dice, colored green, red, and blue, but these dice are not numbered in the standard way. Instead, they are numbered like this:

Green:	5	5	5	2	2	2
Red:	4	4	4	4	4	1
Blue:	3	3	3	3	3	6

In other words, the green die has three faces labeled "5" and three labeled "2"; the red die has five faces labeled "4" and one labeled "1"; and the blue die has five faces labeled "3" and one labeled "6."

Alice proposes to Bob that they play a little game. "Here's how it will work," explains Alice. "In each round, we will each choose one die and discard the third one. Then we will simultaneously roll our chosen dice. Whoever rolls the higher number will win a dollar from the other player. (Note that there can never be a tie.) And since I'm such a nice person, I will let you choose your die first, before I choose mine."

(a) Suppose Bob chooses the red die and Alice chooses the green die. Show that the probability that Alice wins is greater than 1/2.

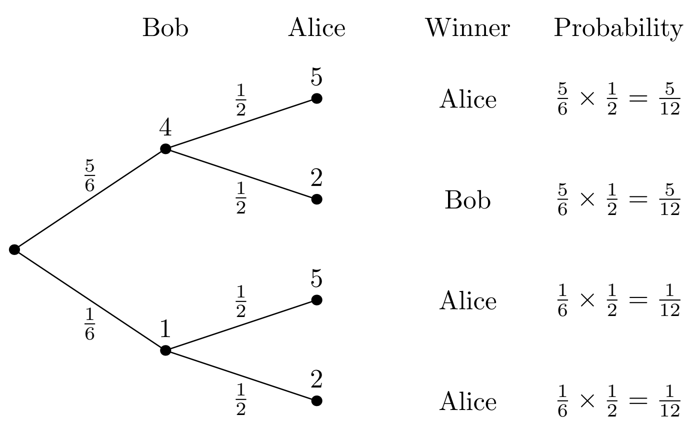
(b) After using the red die for a while and losing more often than not to Alice's green die, Bob begins to suspect that the odds are not in his favor. So he chooses the blue die instead, and Alice chooses the red die. Show that the probability that Alice wins is still greater than 1/2.

(c) Bob continues to lose when he plays the blue die against Alice's red die. He reasons, "The green die is better than the red die, and the red die is better than the blue die. So, clearly, I should choose the green die, because it must be the best of all." He takes the green die, and Alice takes the blue die. Show that the probability that Alice wins is *still* greater than 1/2.

(d) Alice's dice are an example of *nontransitive dice*. How is this dice game similar to the game of Rock, Paper, Scissors?

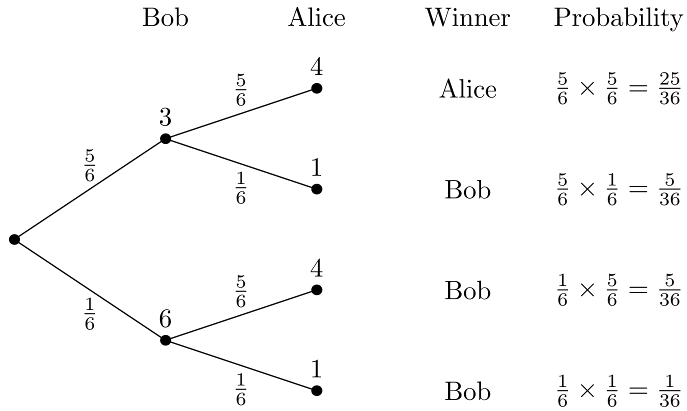
Solution.

(a) We draw a probability tree diagram showing Bob's and Alice's rolls when Bob has the red die and Alice has the green die.



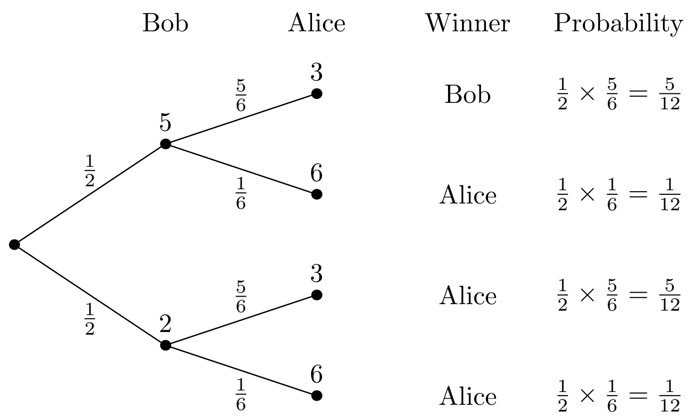
To find the overall probability that Alice wins, we add the probabilities of each outcome in which Alice wins. So the probability that Alice wins is $5/12+1/12+1/12 = 7/12$, which is greater than 1/2 (it is about 58%).

(b) When Bob has the blue die and Alice has the red die, the probability tree diagram looks like the following.



Alice wins with probability 25/36, which is greater than 1/2 (it is about 69%).

(c) When Bob has the green die and Alice has the blue die, the probability tree diagram looks like the following.



Alice wins with probability $1/12 + 5/12 + 1/12 = 7/12$, which is greater than 1/2 (it is about 58%).

(d) In the game of Rock, Paper, Scissors, each object wins over one of the other two objects but loses to the other one: rock beats scissors but loses to paper, paper beats rock but loses to scissors, and scissors beats paper but loses to rock. In Alice's dice game, each die "wins" over one of the other two dice (with probability greater than 1/2) but "loses" to the other one (with probability greater than 1/2): green wins over red but loses to blue, red wins over blue but loses to green, and blue wins over green but loses to red. □

[True story: Warren Buffett once challenged Bill Gates to play this game, offering to let Gates choose his die first. Gates was intrigued, examined the dice, figured out the trick, and then demanded that Buffett choose first.]

Problem 7. Ten mathematicians are captured by pirates. The pirate captain tells them, "I have an unusual custom for dealing with prisoners on my ship. Tomorrow at dawn I will put all ten of you in a line, blindfolded and facing the same direction. I will assign each of you either a black hat or a white hat, depending on the outcome of the flip of a fair coin. (Since there are ten of you, I will flip the coin ten times, once for each of you.)

"After you have each been given a hat, I will remove your blindfolds. Each of you will then be able to see the colors of the hats of all the people standing in front of you in the line, but you will not be able to see your own hat, nor will you be able to see the hats of the people behind you.

"Then I will begin my questioning. I will start with the person at the back of the line (who can see all nine of the other hats). I will ask him what color his hat is. He must announce his guess, either black or white, in a flat, monotone voice, but loudly enough for all the rest of you to hear. If his guess is correct, I will allow him to go free. If his guess is incorrect, however, he will be killed.

"Next, I will move to the second-to-last person in line (who can see eight other hats), and I will repeat the procedure, moving up the line, until I have dealt with each of you in turn. By the way, once the hats have been distributed, you will not be allowed to pass any information among yourselves except your guesses.

"Well, I suppose it's getting pretty late, and we have a big day tomorrow. Sleep well, and I'll see you in the morning!"

That night the mathematicians get together and decide that they must devise a strategy to maximize the expected number of them who will go free. What strategy should they adopt?

Solution. The surprising answer is that nine of the ten mathematicians can be saved with certainty!

Let's number the mathematicians 1 through 10, where mathematician 1 is at the front of the line (and can therefore see no hats) and mathematician 10 is at the back of the line (and can see all nine of the other hats).

The key to the best strategy is the following: Mathematician 10, who is the first to guess, counts the number of white hats he sees in front of him. If he sees an even number of white hats, he guesses "white"; if he sees an odd number of white hats, he guesses "black." Since all of the remaining mathematicians can hear his guess, they will know whether there is an even number or an odd number of white hats among them.

When it is time for mathematician 9 to guess, he can see the number of white hats in front of him, and he knows (from mathematician 10's guess) whether there is an even number or an odd number of white hats among mathematicians 1 through 9. This information is enough to allow mathematician 9 to deduce the color of his hat with certainty. For instance, if he sees an even number of white hats in front of him and he knows that mathematician 10 also saw an even number of white hats, then his own hat must be black. On the other hand, if he sees an even number of white hats in front of him and mathematician 10 saw an *odd* number of white hats, then his own hat must be white. So mathematician 9 can be assured of guessing his own hat color correctly.

Mathematician 8 can use similar reasoning to deduce the color of his hat. When it is his turn to guess, he can see the seven hats in front of him, he knows whether mathematician 10 saw an even number or an odd number of white hats, and he also knows the color of mathematician 9's hat (from mathematician 9's guess, which was correct). This is enough information to allow him to determine his own hat color, so he is also assured of guessing his own hat color correctly.

In general, when it is time for mathematician in the middle of the line to guess, the information available to him includes the colors of all the hats in front of him (which he can see directly); whether mathematician 10 saw an even number or an odd number of white hats (from mathematician 10's guess); and the colors of all the hats behind him, except the hat of mathematician 10 (from the guesses of the other mathematicians). From this information he can work out what color his own hat must be, and so he will be able to guess correctly (simultaneously passing the information about the color of his own hat to the people in front of him, who will need it to determine the colors of their hats).

Therefore, if the mathematicians follow this strategy, mathematicians 1 through 9 will be saved with certainty. Unfortunately, mathematician 10 will only guess correctly half the time (by sheer luck), but, as he has no possible source of information about the color of his own hat, this is the best that can be done. In other words, with probability 1/2 it will be the case that nine mathematicians go free, and with probability 1/2 it will be the case that all ten mathematicians go free. The expected number of survivors is therefore

$$\tfrac{1}{2}(9) + \tfrac{1}{2}(10) = 9.5,$$

which is really quite impressive. □

Problem 8. Suppose the disease diauropunctosis afflicts 1 out of every 5,000 people in the United States. (Assume that the occurrence of the disease is completely random, so that each person has a 1/5,000 probability of having the disease, independently of everyone else.) Fortunately, there is a test for diauropunctosis, which is 99% accurate, meaning that 1% of the people who are tested receive incorrect results. (Assume that the accuracy of the test is independent of the occurrence of diauropunctosis; in other words, assume the test is 99% accurate for people who have the disease and 99% accurate for people who do not have the disease.) The city of Pierce, population 1,000,000, has received a government grant to test all of its citizens for diauropunctosis.

(a) About how many people in Pierce have diauropunctosis?

(b) Out of the people who have diauropunctosis, about how many will test positive for the disease? About how many will test negative?

(c) Out of the people who do *not* have diauropunctosis, about how many will test positive? About how many will test negative?

(d) Out of all the people who test positive for diauropunctosis, what percentage actually have the disease?

(e) Think about what your answer to part (d) means. Suppose you are tested for diauropunctosis, and the test results come back positive. What is the (conditional) probability that you actually have diauropunctosis, given that you tested positive? Explain why receiving a positive result from a test that is 99% accurate does not mean that you have a 99% probability of having the disease.

Solution.

(a) Since the prevalence of diauropunctosis is 1 in 5,000, the number of people in Pierce who have diauropunctosis is about

$$\frac{1}{5{,}000}(1{,}000{,}000) = 200.$$

(b) The test for diauropunctosis is 99% accurate, which means that 99% of the people who are tested will receive the correct result (and the remaining 1% will receive the wrong result). Therefore, out of the 200 people in Pierce who have diauropunctosis, about

$$\begin{aligned} 0.99(200) &= 198 \text{ will test } \textit{positive}; \\ 0.01(200) &= 2 \text{ will test } \textit{negative}. \end{aligned}$$

(c) Out of the 999,800 people in Pierce who do not have diauropunctosis, about

$$\begin{aligned} 0.99(999{,}800) &= 989{,}802 \text{ will test } \textit{negative}; \\ 0.01(999{,}800) &= 9{,}998 \text{ will test } \textit{positive}. \end{aligned}$$

(d) There will be about 198 people who test positive and actually have diauropunctosis, and about 9,998 people who test positive but do not have the disease. So in all there will be about 10,196 people who test positive. Out of these, the percentage who actually have the disease is about

$$\frac{198}{10{,}196} \approx 0.0194 = 1.94\%.$$

(e) The answer to part (d) means that the conditional probability that a person has diauropunctosis, given that the person tested positive, is about 1.94%. The positive test result significantly increased the probability that the person has the disease (from the original probability of 1/5,000 to about 1/51), and so follow-up tests are appropriate, but because of the rarity of the disease it is far more likely that the positive test result was in error than that the person actually has diauropunctosis. The simple truth is that healthy people vastly outnumber people with the disease, and so false positives (though relatively rare with such an accurate test) will still be much more common than actual occurrences of the disease. □