21-110: Problem Solving in Recreational Mathematics

Homework assignment 6

Assigned Friday, March 26, 2010. Due Monday, April 5, 2010.

Work at least **FOUR** of the following problems, at least one of which must be from Part B. All problems are of equal weight. If you submit solutions for more than four problems, you will get credit for your best four (with the proviso that you will get credit for at most three problems from Part A).

You are welcome to work with other students, but the solutions you hand in should be written in your own words. You are not allowed to see the paper another student is going to hand in. If you do collaborate with other students, list their names. If you use other sources, cite them. Give credit where credit is due. See the syllabus for more information about academic integrity.

Hints are encrypted with a *Caesar cipher*, in which each letter is replaced by the letter three places ahead in the alphabet, wrapping around to the beginning if necessary. For example, the letter A is encrypted as D, and Y is encrypted as B. To decrypt the hints, move each letter backward three places.

— Part A —

Problem 1. Normal Pennsylvania license plates have three capital letters followed by four numerical digits. Assuming there are no other restrictions, how many possible license plates can Pennsylvania issue? Why?

Problem 2. In the handout about finding a formula for a sequence of numbers from earlier in the course, we made the conjecture that the number of diagonals of a regular n-gon is

$$\frac{n(n-3)}{2}.$$

Prove this conjecture.

Problem 3. (From *The Colossal Book of Short Puzzles and Problems* by Martin Gardner.) In this country a date such as July 4, 1971, is often written 7/4/71, but in other countries the month is given second and the same date is written 4/7/71. If you do not know which system is being used, how many dates in a year are ambiguous in this two-slash notation?

Problem 4. An artist has painted n landscapes, all different. One of her paintings is unlike all the others, because it shows an unusual white barn. The artist must select r of her landscapes to be displayed in an exhibition. (Assume $r \leq n$.)

(a) In how many ways can she choose r of her n landscapes to be exhibited?

(b) Suppose the artist decides that she likes the painting with the white barn and wants to include it in the exhibition. In how many ways can she choose r of her n landscapes to be exhibited, if one of the r landscapes must be the one with the white barn?

Hint: Vkh kdv douhdgb pdgh rqh fkrlfh. Krz pdqb pruh fkrlfhv pxvw vkh pdnh, dqg krz pdqb odqgvfdshv grhv vkh kdyh wr fkrrvh iurp?

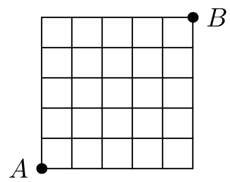
(c) Suppose, on the other hand, that the artist decides she does not like the painting with the white barn and wants to keep it out of the exhibition. In how many ways can she choose r of her n landscapes to be exhibited, if the one with the white barn must not be chosen?

Hint: Krz pdqb fkrlfhv grhv vkh kdyh wr pdnh, dqg krz pdqb odqgvfdshv grhv vkh kdyh wr fkrrvh iurp?

(d) Use the ideas of the previous three parts to prove the general fact that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Problem 5. You find yourself at point A in a city with streets running north–south and east–west that form a square grid (i.e., not Pittsburgh). You need to get to point B, which is five blocks east and five blocks north of point A. (See the street map below.) In how many different ways can you walk from point A to point B, if you always walk only north or east?



Hint: Krz pdqb eorfnv pxvw brx zdon lq doo? Krz pdqb ri wkrvh eorfnv pxvw eh qruwkzdug?

Problem 6. How many five-letter "words" formed from the letters A, B, C, and D (with repetition allowed) contain exactly two A's?

Hint: Rqh zdb wr pdnh vxfk d zrug lv wr iluvw fkrrvh wkh srvlwlrqv ri wkh D'v dqg wkhq fkrrvh wkh uhpdlqlqj ohwwhuv.

Problem 7. How many integers from 1 to 10,000 are divisible by 3 or 7 (or both)?

Hint: Sulqflsoh ri lqfoxvlrq–hafoxvlrq.

— Part B —

Problem 8. (Choosing with replacement, order not important.) Five pirates have decided to divide a treasure of 12 identical gold coins among them. The question we will explore in this problem is: In how many ways can this be done?

(a) One possible way to divide the treasure is as follows: The first pirate gets 3 coins, the second gets 5, the third gets 1, the fourth gets none, and the fifth gets 3. Explain how the following sequence of symbols represents this possibility.

$$\circ\ \circ\ \circ \mid \circ\ \circ\ \circ\ \circ\ \circ \mid \circ \mid \mid \circ\ \circ\ \circ$$

(b) Another possible way to divide the treasure is as follows: The first pirate gets 5 coins, the second gets 1, the third gets 3, the fourth gets 3, and the fifth gets none. [Note that this division of the gold is *different* from the division in part (a).] Write a sequence of symbols similar to the one above representing this possibility.

(c) Consider all such sequences of symbols representing possible ways to divide the gold. What defining characteristics do all such sequences share? Be as precise as you can.

Hint: Lq sduwlfxodu, krz pdqb ri hdfk wbsh ri vbpero?

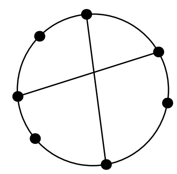
(d) Explain why the number of possible ways to divide the treasure is $\binom{16}{4}$.

Hint: Xvh wkh suhylrxv sduw.

(e) Explain why, in general, if r indistinguishable objects are to be distributed into n distinguishable boxes, there are $\binom{n+r-1}{n-1}$ ways to do so.

(f) In what sense can this be described as "choosing r elements from a set of size n with replacement, when the order in which the elements are chosen is not important"?

Problem 9. Given n points on the circumference of a circle, how many ways are there to draw two different chords connecting pairs of these points such that the chords intersect in the interior of the circle? (Chords sharing one endpoint do not count.) The picture below shows seven points on the circumference of a circle and one way to draw two chords that intersect in the interior.



Problem 10. Use the pigeonhole principle to prove that, if any five points are chosen inside a square whose sides are 2 meters long, there must be two of these points that are no more than $\sqrt{2}$ meters apart.

Hint: Brx duh sodflqj ilyh remhfwv (wkh srlqwv) lqwr irxu "erahv." Zkdw vkrxog wkh "erahv" eh? Wkh Sbwkdjruhdq wkhruhp zloo suredeob eh khosixo dw vrph srlqw.