21-110: Problem Solving in Recreational Mathematics

Homework assignment 4

Assigned Friday, February 19, 2010. Due Friday, February 26, 2010.

Work at least **FOUR** of the following problems. All problems are of equal weight. If you submit solutions for more than four problems, you will get credit for your best four.

You are welcome to work with other students, but the solutions you hand in should be written in your own words. You are not allowed to see the paper another student is going to hand in. If you do collaborate with other students, list their names. If you use other sources, cite them. Give credit where credit is due. See the syllabus for more information about academic integrity.

Hints are encrypted with a *Caesar cipher*, in which each letter is replaced by the letter three places ahead in the alphabet, wrapping around to the beginning if necessary. For example, the letter A is encrypted as D, and Y is encrypted as B. To decrypt the hints, move each letter backward three places.

Problem 1. Recall that $n!$, read "n factorial," is the number $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. The number 110! ends with a bunch of zeroes. How many?

Hint: Wklqn derxw idfwruv. Zkdw zloo fdxvh d chur dw wkh hqg? Zkdw zloo fdxvh wzr churhv dw wkh hqg?

Problem 2. What is the remainder when $3^{21,110}$ is divided by 7?

Hint: Ilqg d sdwwhuq.

Problem 3. In class we showed that the least common multiple of 10 and 12 is $\text{lcm}(10, 12) = 60$ and the greatest common divisor of 10 and 12 is $\gcd(10, 12) = 2$. We then observed that $10 \times 12 = 120$ and $60 \times 2 = 120$. Does this always happen? In other words, is it true that for *every* possible pair of positive integers a and b,

$$a \cdot b = \text{lcm}(a, b) \cdot \gcd(a, b)?$$

If so, explain why. If not, give a counterexample (that is, give two positive integers a and b for which this statement is false).

Hint: Frqvlghu krz wr ilqg wkh ofp dqg wkh jfg iurp wkh sulph idfwrulcdwlrqv ri d dqg e.

Problem 4. A long hallway has 100 light bulbs, numbered 1 through 100. Each light bulb has a pull string; when the string is pulled, the light bulb turns on if it was off, and turns off if it was on. All of the light bulbs are initially off.

One person walks into the hall, pulls every string, and walks out. Then a second person enters, pulls the strings of light bulbs 2, 4, 6, 8, and so on, and leaves. Next, a third person comes in, pulls strings 3, 6, 9, 12, and so on, and exits. This pattern continues until 100 people have walked in, pulled some of the strings, and walked out. (Note that the 100th person will pull only string 100.)

After all 100 people have done this, which light bulbs are on? More importantly, why?

Problem 5. Consider the polynomial function $f(x) = x^2 + x + 41$. Compute the values $f(0)$, $f(1)$, $f(2)$, $f(3)$, ..., $f(20)$. What can you say about the *primality* of these numbers? (That is, are they prime numbers or not?) Does this pattern continue forever? Why or why not?

[This interesting property of this polynomial was discovered by the Swiss mathematician Leonhard Euler (pronounced "Oiler") in 1772.]

Hint: Li brx duh jrlqj wr suryh wkdw wkh sdwwhuq frqwlqxhv iruhyhu, brx qhhg wr hasodlq zkb $i(a)$ pxvw eh sulph iru hyhub qrqqhjdwlyh lqwhjhu a. Li brx duh jrlqj wr suryh wkdw wkh sdwwhuq grhv qrw frqwlqxh iruhyhu, brx qhhg wr ilqg d vlqjoh ydoxh ri a wkdw euhdnv wkh sdwwhuq.

Problem 6. Use the definition of divisibility to prove the following statement about integers a, b, c, x, and y, where $a \neq 0$: If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$.

Problem 7. ("The Cashier's Error," from *The Moscow Puzzles* by Boris A. Kordemsky, edited by Martin Gardner.) The customer said to the cashier: "I have 2 packages of lard at 9 cents; 2 cakes of soap at 27 cents; and 3 packages of sugar and 6 pastries, but I don't remember the prices of the sugar and pastries."

"That will be $2.92."

The customer said: "You have made a mistake."

The cashier checked again and agreed.

How did the customer spot the error?

Problem 8. Use the Euclidean algorithm to find the greatest common divisor of 11,238,781 and 8,476,873.

Hint: Brx fdq fkhfn brxu dqvzhu eb pdnlqj vxuh lw glylghv erwk ri wkh wzr jlyhq qxpehuv.

Problem 9. Investigate the powers of 2 (1, 2, 4, 8, 16, ...) and classify them as abundant, deficient, or perfect. Can you make a conjecture about these numbers? (Make your conjecture as specific as you can.)

Problem 10. (From *Mathematical Recreations and Essays* by W. W. Rouse Ball and H. S. M. Coxeter.) Prove that every sum of two consecutive odd primes is the product of three integers all greater than 1. For example, $7 + 11 = 2 \times 3 \times 3$, and $11 + 13 = 2 \times 3 \times 4$.

[This is not as difficult as it sounds. Start with what you know and see where you can go from there.]

Problem 11. An eccentric sheep farmer has built an enclosure with four separate pens, as shown below. In the center of the enclosure is a rather unusual four-sided gate. Whenever three sheep enter the gate from three different sides, the fourth side of the gate opens and all three sheep enter the fourth pen.

If there are initially 24, 28, 32, and 36 sheep in the four pens, respectively, is it possible that after some time has passed there will be an equal number of sheep in each pen? What if there are initially 12, 14, 16, and 18 sheep in the four pens, respectively? In each case justify your answer by demonstrating how it can be done or by proving that it is impossible.

Hint: Brx pdb ilqg lw xvhixo wr zrun edfnzdugv. Frqvlghu krz wkh jdwh fkdqjhv wkh qxpehu ri vkhhs lq hdfk shq. Lw pdb eh khosixo wr wklqn derxw uhpdlqghuv zkhq fhuwdlq qxpehuv duh glylghg eb irxu.