21-110: Problem Solving in Recreational Mathematics

Homework assignment 4 solutions

Problem 1. Recall that $n!$, read "n factorial," is the number $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. The number 110! ends with a bunch of zeroes. How many?

Solution. A zero at the end of a number is a consequence of a factor of 10. For example, the number 70 ends with one zero because it has one factor of 10 ($70 = 7 \times 10$), and the number 12,000 ends with three zeroes because it has three factors of 10 ($12{,}000 = 12 \times 10 \times 10 \times 10$). So to count the number of zeroes at the end of 110! we need to count the number of factors of 10 in 110!.

One factor of 10 is the same as one factor of 2 and one factor of 5, since 2 and 5 are the prime factors of 10. We can see how many factors of 2 and factors of 5 a number has by looking at the prime factorization of the number.

Let's return to the examples above. The prime factorization of 70 is $2 \times 5 \times 7$. The 2 and the 5 combine to make a factor of 10, so we can tell from the prime factorization that 70 has one zero at the end. Similarly, the prime factorization of 12,000 is $2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 5 \times 5$, or $2^5 \times 3 \times 5^3$. The three factors of 5 combine with three of the five factors of 2 to form three factors of 10 in 12,000, giving us three zeroes at the end. (The other two factors of 2 are "leftovers," because there are no more factors of 5 for them to combine with.)

In a factorial the factors of 2 will always outnumber the factors of 5, since every even number in the product $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$ will contribute at least one factor of 2, whereas factors of 5 are contributed only by multiples of 5 (which occur more rarely). This means that we can count the factors of 10 in a factorial by counting how many factors of 5 it has (because there will always be enough factors of 2 to combine with them).

In the number 110!, every multiple of 5 from 5 up to 110 contributes one factor of 5. This gives us $110/5 = 22$ factors of 5, which will give us 22 zeroes at the end of 110!.

But there are a few more factors of 5. The number 25 will contribute not one but two factors of 5 to the number 110!, as will all multiples of 25 (50, 75, and 100). So there are four additional factors of 5, giving us four more zeroes at the end of 110!. (If, instead of 110!, we had been considering the number 150!, say, we would also need to consider the fact that 5^3, or 125, will contribute *three* factors of 5, and so on.)

In total, then, the number 110! ends with 26 zeroes. □

Problem 2. What is the remainder when $3^{21{,}110}$ is divided by 7?

Solution. There are at least three ways to solve this problem. Perhaps the simplest is to start looking at small powers of 3 and their remainders when divided by 7, in the hopes of finding a pattern.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	...
3^n	3	9	27	81	243	729	2,187	6,561	19,683	59,049	177,147	531,441	1,594,323	...
rem.	3	2	6	4	5	1	3	2	6	4	5	1	3	...

We see that these remainders repeat 3, 2, 6, 4, 5, 1 over and over, in other words, every sixth power of 3. The greatest multiple of 6 less than 21,110 is 21,108, which is $6 \times 3{,}518$. So if we were to continue this table all the way to $3^{21{,}110}$, we would have 3,518 repetitions of the sequence 3, 2, 6, 4, 5, 1, which would bring us up to $3^{21{,}108}$; two more powers of 3 would give us $3^{21{,}110}$ and would bring us two numbers into the next repetition of the sequence. The second number of the repeating sequence is 2, so the remainder when $3^{21{,}110}$ is divided by 7 is 2.

The second approach is similar, but uses modular arithmetic to simplify the calculations. Since we are interested in remainders when numbers are divided by 7, we can do all of our calculations modulo 7 and work only with remainders rather than having to juggle big numbers like 1,594,323. Obviously $3^1 \equiv 3 \pmod 7$. Next we multiply 3^1 by 3 and get $3 \times 3 = 9$, which has a remainder of 2 when divided by 7; so $3^2 \equiv 2 \pmod 7$. To find 3^3, rather than multiplying 9 by 3, we can simply multiply this remainder 2 by 3 to get $3^3 = 3^2 \times 3 \equiv 2 \times 3 = 6 \pmod 7$. For 3^4, we multiply this remainder 6 by 3 to get $3^4 = 3^3 \times 3 \equiv 6 \times 3 = 18 \equiv 4 \pmod 7$. Continuing in this way, we can build up a table as before, calculating only with remainders:

n	1	2	3	4	5	6	7	...
$3^n \pmod 7$	3	2	6	4	5	1	3	...

We don't have to extend the table as far to see the pattern this time, because we know (from the way we are calculating these remainders) that the pattern will repeat once we've come back to where we started. For example, to find 3^8, we multiply 3^7 (which is congruent to 3 modulo 7) by 3 to get $3^8 = 3^7 \times 3 \equiv 3 \times 3 = 9 \equiv 2 \pmod 7$, which is the same as 3^2.

Now that we have found the pattern, we can use the same reasoning as we used in the first approach to see that the pattern of remainders will fully repeat 3,518 times, and go two numbers further, giving us a remainder of 2 when $3^{21,110}$ is divided by 7.

The third approach is a further refinement. We begin this time by finding the smallest positive integer power of 3 that is congruent to 1 modulo 7. So we start with 3^1 and then find 3^2, 3^3, and so on (using modular arithmetic) until we get a remainder of 1. From what we have done before, we know that 6 is the smallest such power of 3, because $3^6 \equiv 1 \pmod 7$. We can use this fact to compute the remainder when $3^{21,110}$ is divided by 7 by using modular arithmetic (modulo 7) and grouping the factors of 3 into clusters of six, each of which is congruent to 1 modulo 7:

$$\begin{aligned}
3^{21,110} &= \underbrace{3 \times 3 \times 3 \times \cdots \times 3}_{\text{21,110 factors}} \\
&= \underbrace{(3{\times}3{\times}3{\times}3{\times}3{\times}3) \times (3{\times}3{\times}3{\times}3{\times}3{\times}3) \times \cdots \times (3{\times}3{\times}3{\times}3{\times}3{\times}3)}_{\text{3,518 of these clusters}} \times \underbrace{3 \times 3}_{\text{leftovers}} \\
&\equiv \underbrace{1 \times 1 \times \cdots \times 1}_{\text{3,518 factors}} \times 3 \times 3 \pmod 7 \\
&= 9 \\
&\equiv 2 \pmod 7.
\end{aligned}$$

Any way we do it, we see that $3^{21,110}$ yields a remainder of 2 when it is divided by 7. □

Problem 3. In class we showed that the least common multiple of 10 and 12 is $\operatorname{lcm}(10, 12) = 60$ and the greatest common divisor of 10 and 12 is $\gcd(10, 12) = 2$. We then observed that $10 \times 12 = 120$ and $60 \times 2 = 120$. Does this always happen? In other words, is it true that for *every* possible pair of positive integers a and b,

$$a \cdot b = \operatorname{lcm}(a, b) \cdot \gcd(a, b)?$$

If so, explain why. If not, give a counterexample (that is, give two positive integers a and b for which this statement is false).

Solution. This equality is always true, for every possible pair of positive integers a and b. To see why this is, we consider the prime factorizations of a and b.

Suppose p is a prime number that is a factor of a or b (or both). Suppose further that a has exactly m factors of p (that is, $p^m \mid a$ but $p^{m+1} \nmid a$) and that b has exactly n factors of p. (It is possible that one of m or n is zero, if p is not a factor of both a and b.) Then their product ab has exactly $m + n$ factors of p.

How many factors of p are in $\operatorname{lcm}(a, b)$ and $\gcd(a, b)$? To find the greatest common divisor of a and b, we take as many factors of p as possible without exceeding m or n. In other words, if $m < n$ then $\gcd(a, b)$ has exactly m factors of p, and if $m \geq n$ then $\gcd(a, b)$ has exactly n factors of p.

Similarly, to find the least common multiple of a and b, we take the smallest number of factors of p that is at least as large as both a and b. So if $m < n$ then $\text{lcm}(a, b)$ has exactly n factors of p, and if $m \geq n$ then $\text{lcm}(a, b)$ has exactly m factors of p.

Therefore we have two cases. In the first case m is less than n, so $\text{gcd}(a, b)$ has exactly m factors of p, and $\text{lcm}(a, b)$ has exactly n factors of p. Then the product $\text{lcm}(a, b) \cdot \text{gcd}(a, b)$ has exactly $m + n$ factors of p. In the second case m is greater than or equal to n, so $\text{gcd}(a, b)$ has exactly n factors of p, and $\text{lcm}(a, b)$ has exactly m factors of p, meaning that the product $\text{lcm}(a, b) \cdot \text{gcd}(a, b)$ again has exactly $m + n$ factors of p.

Hence, in any case, $\text{lcm}(a, b) \cdot \text{gcd}(a, b)$ has exactly as many factors of p as ab does. Since p represents an arbitrary prime factor, this argument must remain true for *all* the prime factors of either a or b. Therefore the prime factorization of $\text{lcm}(a, b) \cdot \text{gcd}(a, b)$ is exactly the same as the prime factorization of ab, which means that $a \cdot b = \text{lcm}(a, b) \cdot \text{gcd}(a, b)$. □

Problem 4. A long hallway has 100 light bulbs, numbered 1 through 100. Each light bulb has a pull string; when the string is pulled, the light bulb turns on if it was off, and turns off if it was on. All of the light bulbs are initially off.

One person walks into the hall, pulls every string, and walks out. Then a second person enters, pulls the strings of light bulbs 2, 4, 6, 8, and so on, and leaves. Next, a third person comes in, pulls strings 3, 6, 9, 12, and so on, and exits. This pattern continues until 100 people have walked in, pulled some of the strings, and walked out. (Note that the 100th person will pull only string 100.)

After all 100 people have done this, which light bulbs are on? More importantly, why?

Solution. The first observation we make is that, at the end, the light bulbs that are on are precisely those light bulbs whose strings were pulled an *odd* number of times. This is a useful observation to make, even though it doesn't immediately go very far in telling us which of the numbered light bulbs will be on.

From some experimentation (perhaps considering only the first 25 light bulbs rather than all 100), we see that the light bulbs that are on at the end are numbers 1, 4, 9, 16, 25, and so on. This leads us to conjecture that it is the perfect squares that will be on.

Let's consider a given string (say, the string for light bulb 12). Which people pull this string? Well, string 12 is pulled by people 1, 2, 3, 4, 6, and 12. Aha! Since each person pulls the strings that are multiples of his or her number, each string will be pulled once for each of its divisors. Since 12 has six divisors (including 12 itself), the string for light bulb 12 will be pulled six times. Since 6 is an even number, this means that light bulb 12 will be off.

From the observation we made at the start, we see that our conjecture is equivalent to the following statement:

Every perfect square has an odd number of divisors;
every number that is not a perfect square has an even number of divisors.

If we can establish that this statement is true, we will have an explanation for the claim that the light bulbs that are on are precisely those whose numbers are perfect squares.

Let's begin with the claim that every number that is not a perfect square has an even number of divisors. (Let's continue using 12 as our example.) There is a trick that is often useful when trying to show that some collection has an even number of things: Try to find a way to pair them up. Is there a way that we can pair up the divisors of 12? After some thought, we see that there is indeed a very natural way to do this. For every divisor n of 12, the number $12/n$ is also a divisor of 12. So the divisors 1 and 12 can be paired up, because $1 \times 12 = 12$; 2 and 6 can be paired up, because $2 \times 6 = 12$; and 3 and 4 can be paired up, because $3 \times 4 = 12$. Since we are able to pair up all of the divisors of 12 with no leftovers, we see that 12 has an even number of divisors. It seems that this trick will work with the divisors of any positive integer, which would show that every positive integer has an even number of divisors.

But wait. We are also claiming that every perfect square has an odd number of divisors, so this pairing-up trick must fail for the divisors of a perfect square. Why? Let's consider 16 as an example. The divisors of 16 are 1, 2, 4, 8, and 16. If we try to pair them up, we put 1 with 16, because $1 \times 16 = 16$; we put 2 with 8, because $2 \times 8 = 16$; and we put 4 with—oops. We would like to put 4 with 4 (because $4 \times 4 = 16$), but then we aren't *pairing* it with anything. This is the reason a perfect square has an odd number of divisors: each of the divisors pairs up with some other divisor, *except* the square root. Since we have one divisor left over after this pairing-up, a perfect square must have an odd number of divisors.

Therefore, our final answer (and justification) goes like this: Every positive integer a that is not a perfect square has an even number of divisors, because every divisor n of a can be paired with the divisor a/n, which is an integer (since $n \mid a$) and is different from n (since a is not a perfect square). On the other hand, every perfect square b has an odd number of divisors, because every divisor of b other than the square root can be paired with another divisor of b as above, but the square root $\sqrt{b}$ is left over at the end.

Now, the string for a given light bulb will be pulled once for each of the divisors of the number on the bulb, and at the end a light bulb will be on if and only if its string was pulled an odd number of times. Therefore, the light bulbs that are on after all 100 people have gone through the hall are precisely those bulbs marked with a perfect square. $\square$

[See also "Odd Divisors" on page 184 of *Thinking Mathematically*.]

Problem 5. Consider the polynomial function $f(x) = x^2 + x + 41$. Compute the values $f(0)$, $f(1)$, $f(2)$, $f(3)$, ..., $f(20)$. What can you say about the *primality* of these numbers? (That is, are they prime numbers or not?) Does this pattern continue forever? Why or why not?

[This interesting property of this polynomial was discovered by the Swiss mathematician Leonhard Euler (pronounced "Oiler") in 1772.]

Solution. The values of $f(x)$ for the first 40 nonnegative integer values of x are given below.

$f(0) = 41$, prime.	$f(10) = 151$, prime.	$f(20) = 461$, prime.	$f(30) = 971$, prime.
$f(1) = 43$, prime.	$f(11) = 173$, prime.	$f(21) = 503$, prime.	$f(31) = 1033$, prime.
$f(2) = 47$, prime.	$f(12) = 197$, prime.	$f(22) = 547$, prime.	$f(32) = 1097$, prime.
$f(3) = 53$, prime.	$f(13) = 223$, prime.	$f(23) = 593$, prime.	$f(33) = 1163$, prime.
$f(4) = 61$, prime.	$f(14) = 251$, prime.	$f(24) = 641$, prime.	$f(34) = 1231$, prime.
$f(5) = 71$, prime.	$f(15) = 281$, prime.	$f(25) = 691$, prime.	$f(35) = 1301$, prime.
$f(6) = 83$, prime.	$f(16) = 313$, prime.	$f(26) = 743$, prime.	$f(36) = 1373$, prime.
$f(7) = 97$, prime.	$f(17) = 347$, prime.	$f(27) = 797$, prime.	$f(37) = 1447$, prime.
$f(8) = 113$, prime.	$f(18) = 383$, prime.	$f(28) = 853$, prime.	$f(38) = 1523$, prime.
$f(9) = 131$, prime.	$f(19) = 421$, prime.	$f(29) = 911$, prime.	$f(39) = 1601$, prime.

Surely, with this much evidence, it seems a safe bet that $f(x)$ is prime for all nonnegative integer values of x. But this is not the case! Let's go just two steps further and consider $f(41)$:

$$f(41) = 41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \times 43,$$

so $f(41)$ is composite. Therefore the answer to the question is no; the pattern continues for quite a while, but not forever. In fact there are infinitely many values of x for which $f(x)$ is composite (any multiple of 41, for example).

It turns out that $f(40)$ is also composite, although this takes some cleverer algebraic factoring to see:

$$f(40) = 40^2 + 40 + 41 = 40^2 + 40 + 40 + 1 = 40^2 + 2(40) + 1 = (40 + 1)^2 = 41^2 = 41 \times 41. \qquad \square$$

[This is a famous example of a false pattern that strongly appears to be true if only the first few cases are investigated. Examples such as these emphasize the importance of mathematical *proof*, rather than simply strong evidence, to establish that conjectures are true.]

Problem 6. Use the definition of divisibility to prove the following statement about integers a, b, c, x, and y, where $a \neq 0$: If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$.

Solution. Since $a \mid b$, there exists an integer d such that $b = ad$ (by the definition of divisibility). Likewise, since $a \mid c$, there exists an integer e (probably different from d) such that $c = ae$. Therefore,

$$bx + cy = (ad)x + (ae)y = a(dx) + a(ey) = a(dx + ey).$$

Since $dx + ey$ is an integer, we see from the definition of divisibility that $a \mid (bx + cy)$. □

Problem 7. ("The Cashier's Error," from *The Moscow Puzzles* by Boris A. Kordemsky, edited by Martin Gardner.) The customer said to the cashier: "I have 2 packages of lard at 9 cents; 2 cakes of soap at 27 cents; and 3 packages of sugar and 6 pastries, but I don't remember the prices of the sugar and pastries."

"That will be \$2.92."

The customer said: "You have made a mistake."

The cashier checked again and agreed.

How did the customer spot the error?

Solution. The prices of the lard and soap are both multiples of 3 cents, so the total price of the lard and soap is also a multiple of 3 cents. The quantities of the sugar and pastries are both multiples of 3, so the total price of the sugar and pastries must be a multiple of 3 cents. Therefore, the total of everything at the end should also be a multiple of 3 cents. But it's not, because 292 is not a multiple of 3 (using the rule for divisibility by 3: $2 + 9 + 2 = 13$ is not a multiple of 3, so neither is 292). Therefore, the cashier must have made an error. □

[This solution uses the "two out of three theorem," Theorem 4.1(iii) on page 105 of *Problem Solving Through Recreational Mathematics*, three times.]

Problem 8. Use the Euclidean algorithm to find the greatest common divisor of 11,238,781 and 8,476,873.

Solution. The process of the Euclidean algorithm is presented below in a table, in the same way as the example in the "Additional topics in number theory" handout (available online at `http://www.math.cmu.edu/~bkell/21110-2010s/numbers.html`).

	c	d	r
Step 1	11,238,781	8,476,873	
Step 2	11,238,781	8,476,873	2,761,908
Step 4	8,476,873	2,761,908	
Step 2	8,476,873	2,761,908	191,149
Step 4	2,761,908	191,149	
Step 2	2,761,908	191,149	85,822
Step 4	191,149	85,822	
Step 2	191,149	85,822	19,505
Step 4	85,822	19,505	
Step 2	85,822	19,505	7,802
Step 4	19,505	7,802	
Step 2	19,505	7,802	3,901
Step 4	7,802	3,901	
Step 2	7,802	3,901	0
Step 3	gcd(11,238,781, 8,476,873) = 3,901		

Therefore the greatest common divisor of 11,238,781 and 8,476,873 is 3,901. □

Problem 9. Investigate the powers of 2 (1, 2, 4, 8, 16, . . .) and classify them as abundant, deficient, or perfect. Can you make a conjecture about these numbers? (Make your conjecture as specific as you can.)

Solution. A table listing the first eight powers of 2, their proper divisors, the sums of their proper divisors, and their classification as abundant, deficient, or perfect is given below.

Number	Proper divisors	Sum	Classification
1	*(none)*	0	deficient
2	1	1	deficient
4	1, 2	3	deficient
8	1, 2, 4	7	deficient
16	1, 2, 4, 8	15	deficient
32	1, 2, 4, 8, 16	31	deficient
64	1, 2, 4, 8, 16, 32	63	deficient
128	1, 2, 4, 8, 16, 32, 64	127	deficient

From this table we can make several conjectures.

Conjecture 1. *The set of proper divisors of a power of 2 is the set of all smaller powers of 2.*

Conjecture 2. *The sum of the proper divisors of a power of 2 is always one less than the power of 2 itself.*

Conjecture 3. *Every power of 2 is deficient by exactly 1.*

(Note that Conjecture 3 is just a rewording of Conjecture 2.) □

[In fact all three of these conjectures are true. Conjecture 1 can be proved by considering the prime factorization of a power of 2 (can you see how?). One way to prove Conjecture 2 is by using a technique called *mathematical induction*, which we may talk about later in the course. And, of course, if Conjecture 2 is true then Conjecture 3 must be true also.]

Problem 10. (From *Mathematical Recreations and Essays* by W. W. Rouse Ball and H. S. M. Coxeter.) Prove that every sum of two consecutive odd primes is the product of three integers all greater than 1. For example, $7 + 11 = 2 \times 3 \times 3$, and $11 + 13 = 2 \times 3 \times 4$.

Solution. Let p and q be two consecutive odd primes. (In other words, p and q are both odd integers, they are both primes, and every integer strictly between p and q is composite.) Our goal is to show that $p + q$ can be written as the product of three integers all greater than 1.

Since p and q are both odd, their sum $p+q$ is even. Therefore, $p+q$ is divisible by 2, so $(p+q)/2$ is an integer. But $(p + q)/2$ is the *average* of p and q, which must be a number strictly between p and q (because $p \neq q$). Hence $(p+q)/2$ must be composite (because p and q are consecutive primes). So we can write $(p + q)/2$ as the product of two integers both greater than 1, say, $(p + q)/2 = mn$.

Thus, we can write the sum $p + q$ as

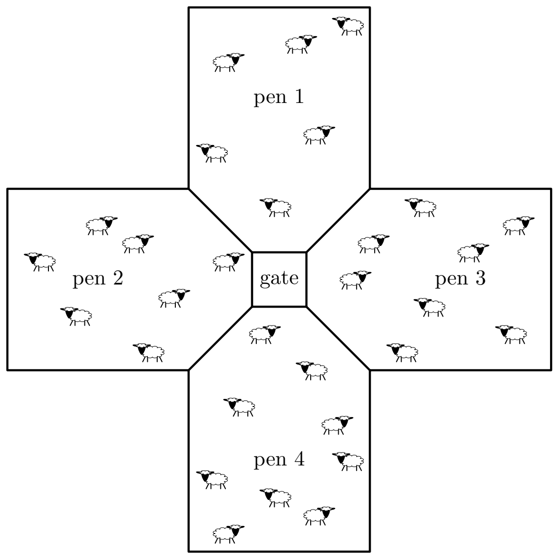
$$p + q = 2\left(\frac{p+q}{2}\right) = 2(mn),$$

so $p + q$ is the product of 2, m, and n, each of which is an integer greater than 1. □

[Note that this proof does not use the fact that p and q are themselves prime; it requires only that p and q are both odd and that every integer between p and q is composite.]

Problem 11. An eccentric sheep farmer has built an enclosure with four separate pens, as shown below. In the center of the enclosure is a rather unusual four-sided gate. Whenever three sheep enter the gate from three different sides, the fourth side of the gate opens and all three sheep enter the fourth pen.

If there are initially 24, 28, 32, and 36 sheep in the four pens, respectively, is it possible that after some time has passed there will be an equal number of sheep in each pen? What if there are initially 12, 14, 16, and 18 sheep in the four pens, respectively? In each case justify your answer by demonstrating how it can be done or by proving that it is impossible.



Solution. Suppose there are initially 24, 28, 32, and 36 sheep in the four pens, so there are 120 sheep in all. If there are to be an equal number of sheep in each of the four pens, there should be 30 sheep in each. One of the pens must lose 6 sheep for this to happen. Each use of the gate can at best decrease the number of sheep in this pen by 1, so we will need to use the gate at least 6 times.

If our goal is to equalize the number of sheep in each pen, one sensible strategy seems to be the following: Find the pen with the least number of sheep (breaking ties arbitrarily), and move into it one sheep from each of the other three pens. Repeat until all pens have the same number of sheep.

On the face of it, there seems to be no particular reason that this strategy should always work; but if we try it on this specific example, it does work, and more impressively it is able to achieve our goal with the minimum of 6 uses of the gate, as shown below.

24	28	32	36
+ 3	− 1	− 1	− 1
27	27	31	35
+ 3	− 1	− 1	− 1
30	26	30	34
− 1	+ 3	− 1	− 1
29	29	29	33
+ 3	− 1	− 1	− 1
32	28	28	32
− 1	+ 3	− 1	− 1
31	31	27	31
− 1	− 1	+ 3	− 1
30	30	30	30

Now suppose there are initially 12, 14, 16, and 18 sheep in the four pens, so there are 60 sheep in all. Our goal is to get 15 sheep in each pen. When we try to use the previous strategy, we are repeatedly frustrated; it seems there is no way to do it. Can we *prove* that it is impossible?

Here is one such proof, using modular arithmetic. We shall consider the remainders when the number of sheep in each pen is divided by 4. For brevity, we shall simply say "remainders" when we mean the remainders after the number of sheep in each pen is divided by 4.

Let us consider how one use of the gate affects the remainders. Three of the remainders are decreased by 1 (because three of the pens lose a sheep), while the other remainder is increased by 3. But, modulo 4, adding 3 is the same as subtracting 1! For example, $2 - 1 \equiv 1 \pmod 4$ and $2 + 3 \equiv 1 \pmod 4$. So each use of the gate affects all four remainders the same way. If the four remainders were equal before the gate was used, they will be equal after; and if they were unequal before, they will be unequal after.

Initially we have 12, 14, 16, and 18 sheep in the four pens, so the remainders start at 0, 2, 0, and 2. In particular, the starting remainders are *unequal*.

Our goal is to have an equal number of sheep in each pen; specifically, we want to have 15 sheep in each pen. Since $15 \equiv 3 \pmod 4$, the remainders will all be 3. The important thing is that, in our goal, all of the remainders are *equal*.

Since a use of the gate cannot change unequal remainders to equal remainders, there is no way to achieve our goal of an equal number of sheep in each pen if we initially have 12, 14, 16, and 18 sheep in the four pens. $\square$