21-110: Problem Solving in Recreational Mathematics

Homework assignment 3

Assigned Friday, February 5, 2010. Due Friday, February 12, 2010.

Work at least **FOUR** of the following problems. All problems are of equal weight. If you submit solutions for more than four problems, you will get credit for your best four.

You are welcome to work with other students, but the solutions you hand in should be written in your own words. You are not allowed to see the paper another student is going to hand in. If you do collaborate with other students, list their names. If you use other sources, cite them. Give credit where credit is due. See the syllabus for more information about academic integrity.

Hints are encrypted with a *Caesar cipher*, in which each letter is replaced by the letter three places ahead in the alphabet, wrapping around to the beginning if necessary. For example, the letter A is encrypted as D, and Y is encrypted as B. To decrypt the hints, move each letter backward three places.

Problem 1. ("A Swimmer and a Hat," from *The Moscow Puzzles* by Boris A. Kordemsky, edited by Martin Gardner.) A boat is being carried away by a current. A man jumps out and swims against the current for a while, then turns around and catches up with the boat. Did he spend more time swimming against the current or catching up with the boat? (We assume his muscular efforts never change in strength.)

The answer is: Both times were the same. The current carries man and boat downstream at the same speed. It does not affect the distance between the swimmer and the boat.

Now imagine that a sportsman jumps off a bridge and begins to swim against the current. The same moment a hat blows off a man's head on the bridge and begins to float downstream. After 10 minutes the swimmer turns back, reaches the bridge, and is asked to swim on until he catches up with the hat. He does, under a second bridge 1,000 yards from the first.

The swimmer does not vary his effort. What is the speed of the current?

Hint: Wkhuh lv d zdb wr vroyh wklv sureohp zlwk doprvw qr dojheud.

Problem 2. International paper sizes, standardized in ISO 216, are used in nearly every country in the world except the United States and Canada. The largest size in the so-called "A series" is called A0, and has an area of one square meter. The next size is called A1 and is formed by cutting a sheet of A0 paper in half. Then A2 paper is formed by cutting a sheet of A1 paper in half, and so on. This continues through the smallest size, A10. Additionally, the various sizes are defined so that all of the paper sizes are similar (that is, they have the same shape, but different sizes). This makes it easy to scale documents from one size of paper to another. What size of paper corresponds most closely to the "letter" size paper common in the United States ($8\frac{1}{2}$ inches by 11 inches)? What are its dimensions, rounded to the nearest millimeter? [There are 25.4 millimeters in one inch, and 1000 millimeters in one meter.]

Hint: Gudz d slfwxuh. Wklqn derxw wkh *sursruwlrqv* ri wkh sdshu. Zkdw grhv lw phdq iru wzr uhfwdqjohv wr eh vlplodu?

Problem 3. (Exercise 3.27 from *Problem Solving Through Recreational Mathematics*.) When Erica was two years old, Leroy was four times as old as Miriam. When Miriam was twice as old as Erica, Leroy was three times as old as Miriam. How old was Erica when Leroy was twice as old as Miriam?

Hint: Wklv sureohp uhihuv wr wkuhh gliihuhqw srlqwv lq wlph. Lw khoshg ph wr gudz d wlpholqh zlwk wkuhh srlqwv pdunhg vr L frxog odeho zkdw zdv wuxh dqg zkhq. Li brx lqwurgxfh yduldeohv, eh vxuh wr fodulib hadfwob zkdw brx lqwhqg iru wkhp wr phdq.

Problem 4. (Exercise 3.19 from *Problem Solving Through Recreational Mathematics.*) The silver currency of the Kingdom of Bonoria consists of glomeks, nindars, and morms. Four glomeks are equal in value to seven nindars; and one glomek and one nindar together are worth thirty-three morms.

On my last visit to Bonoria, I entered a bank, handed the teller some glomeks and nindars, and asked him to change them into morms.

"Do you think that I am a magician?" he replied. (Bonorians are noted for their warped sense of humor.) "Well, let's see," he continued. "If you had twice as many glomeks, I could give you 120 morms; and if you had twice as many nindars I could give you 114 morms."

How many morms did he give me?

Hint: Lw pljkw eh khosixo iluvw wr iljxuh rxw krz pdqb prupv d jorphn lv zruwk, dqg krz pdqb prupv d qlqgdu lv zruwk. Brx pdb kdyh pdqb gliihuhqw yduldeohv lq brxu vhwxs; zulwh grzq suhflvhob zkdw hdfk ri wkhp vwdqgv iru. Wkh "D" eb wkh sureohp qxpehu lq wkh errn lqglfdwhv wkdw wkh dqvzhu (exw qrw wkh vroxwlrq) fdq eh irxqg lq wkh dqvzhuv vhfwlrq ehjlqqlqj rq sdjh 433.

Problem 5. (Exercise 3.63 from *Problem Solving Through Recreational Mathematics*; originally from the *Greek Anthology*, compiled about A.D. 500 by Metrodorus.) I am a brazen lion, a fountain; my spouts are my two eyes, my mouth, and the flat of my right foot. My right eye fills a jar in two days [1 day = 12 hours], my left eye in three, and my foot in four; my mouth is capable of filling it in six hours. Tell me how long all four together will take to fill it.

Hint: Wklv lv vlplodu wr Vdpsoh Sureohp 3.5.

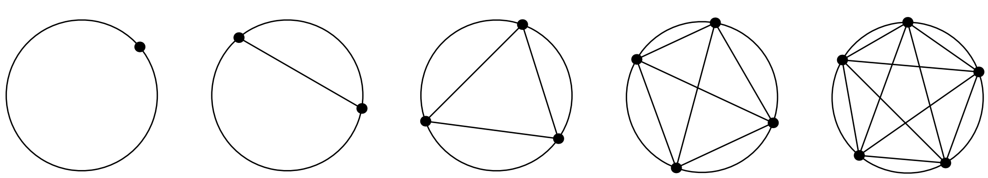
Problem 6. There is a unique real number x that can be expressed in the following form:

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}},$$

where the dots "$\cdots$" mean "and so on, forever." What is this number x? (This strange-looking kind of infinite fraction is called a "continued fraction.")

Hint: Zkdw lv rqh soxv wkh uhflsurfdo ri a? Qrwh wkdw a pxvw eh srvlwlyh, ehfdxvh hyhubwklqj wkdw lv ehlqj dgghg lv srvlwlyh.

Problem 7. In the figures below, there are n points on the circumference of a circle, and a chord is drawn between every pair of points. This divides the circle into a number of regions. The points are chosen in such a way that no three of the chords intersect in a single point, so that the number of regions is maximized. How many regions would be formed if 20 points were chosen around the circumference in this way? (*Be careful*—the "obvious" pattern in the number of regions in the examples below does *not* hold in general! You will need to count the number of regions for $n = 6$, and probably $n = 7$, in order to find the general pattern.)



$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1 region	2 regions	4 regions	8 regions	16 regions

Hint: Ilqg d srobqrpldo irupxod iru wkh qxpehu ri uhjlrqv lq whupv ri q. Wub wkh phwkrg zh hasoruhg lq fodvv, lq zklfk zh orrnhg dw wkh gliihuhqfhv ehwzhhq vxffhvvlyh qxpehuv lq d vhtxhqfh.

Problem 8. (From *Challenging Problems in Algebra* by Alfred S. Posamentier and Charles T. Salkind.) A shopkeeper orders 19 large and 3 small packets of marbles, all alike. When they arrive at the shop, he finds the packets broken open with all the marbles loose in the container. Can you help the shopkeeper make new packets with the proper number of marbles in each, if the total number of marbles is 224?

Hint: Wklqn derxw zkdw nlqg ri qxpehuv zrxog pdnh vhqvh iru wkh vroxwlrq lq wkh frqwhaw ri wkh sureohp.

Problem 9. (From a short story, "Coconuts," by Ben Ames Williams.) . . . So at last Wadlin told him. "Well," he explained, "according to the way the thing was given to me, five men and a monkey were shipwrecked on a desert island, and they spent the first day gathering coconuts for food. Piled them all up together and then went to sleep for the night.

"But when they were all asleep one man woke up, and he thought there might be a row about dividing the coconuts in the morning, so he decided to take his share. So he divided the coconuts into five piles. He had one coconut left over, and he gave that to the monkey, and he hid his pile and put the rest all back together."

He looked at Marr; the man was listening attentively.

"So by and by the next man woke up and did the same thing," Wadlin continued. "And he had one left over, and he gave it to the monkey. And all five of the men did the same thing, one after the other, each one taking a fifth of the coconuts in the pile when he woke up, and each one having one left over for the monkey. And in the morning they divided what coconuts were left, and they came out in five equal shares."

He added morosely, "Of course each one must have known there were coconuts missing; but each one was guilty as the others, so they didn't say anything."

Marr asked sharply, "But what's the question?"

"How many coconuts were there in the beginning?" Wadlin meekly explained.

Hint: Wklv lv d kdug sxccoh! Grq'w vshqg wrr pxfk wlph rq lw xqohvv brx kdyh douhdgb zrunhg rxw irxu ri wkh rwkhu sureohpv. Wkhuh duh dfwxdoob lqilqlwhob pdqb vroxwlrqv; wkh txhvwlrq lv suredeob dvnlqj iru wkh vpdoohvw srvlwlyh qxpehu ri frfrqxwv wkdw zrxog vdwlvib wkh frqglwlrqv.