Department of Mathematical Sciences
Carnegie Mellon University

21-366 Random Graphs
Test 1

You can use my book and you can quote theorems from the book.

Problem	Points	Score
1	30	
2	30	
3	40	
Total	100	

Q1: (30pts)
Suppose that $p = n^{-3/5}$ and we randomly color the edges of $G_{n,p}$ with two colors, Red and Blue. Show that w.h.p. there is a Red-Blue-Red path in $G_{n,p}$ between every pair of vertices.
Hint: think diameter, not second moment.
Solution: Fix two vertices i, j. The red degree of vertex i is distributed as $Bin(n-1, p/2)$. This has expectation $(n^{2/5}-p)/2$ and so the Chernoff bounds imply that with probability $e^{-\Omega(n^{2/5})}$ it has a red degree in $I = [n^{2/5}/4, n^{2/5}]$. Condition on the degree of i being in I. Now consider the number of neighbors of j that are not neighbors of i. This is distributed as $Bin(n - O(n^{2/5}), p/2)$ and so the Chernoff bounds imply that with probability $e^{-\Omega(n^{2/5})}$ it has a red degree in I. It follows that

$$\Pr(\text{there is no RBR path from } i \text{ to } j) \leq e^{-\Omega(n^{2/5})} + \left(1 - \frac{p}{2}\right)^{n^{4/5}/16} = o(n^{-2}).$$

Now use the union bound over choices of i, j.

Q2: (30pts)
Suppose that $0 < \epsilon$ is a small constant and that $\frac{\alpha(1-\log\alpha)}{1-\alpha} < \epsilon$. Show that if $p = \frac{(1+\epsilon)\log n}{n}$ then w.h.p. the minimum degree in $G_{n,p}$ is at least $\alpha \log n$.
Hint: no vertices of degree less than $\alpha \log n$.
Solution: Let X denote the number of vertices of degree less than $a = \alpha \log n$. Then,

$$\begin{aligned}
\mathbf{E}(X) &= n \sum_{k=0}^{a} \binom{n-1}{k} p^k (1-p)^{n-1-k} \\
&\leq n \sum_{k=0}^{a} \left(\frac{ne}{k}\right)^k \left(\frac{(1+\epsilon)\log n}{n}\right)^k n^{-(1+\epsilon+o(1))} \\
&= n \sum_{k=0}^{a} \left(\frac{e(1+\epsilon)\log n}{k}\right)^k n^{-(1+\epsilon+o(1))}.
\end{aligned}$$

Let $f(x) = (eA/x)^x$. Then $f'(x) = f(x)\log(A/x)$ and so f increases from

$x = 0$ to $x = A$ and $f(A) = e^A$. So,

$$\mathbf{E}(X) \leq \alpha n \log n \times n^{a\log(e(1+\epsilon)/a)-(1+\epsilon+o(1))} = n^{a(1+\log(1+\epsilon)/a)-\epsilon+o(1)} \leq$$
$$n^{a(1+\epsilon-\log a)-\epsilon+o(1)} = o(1).$$

Q3: (40pts)
Suppose that $p = \frac{\omega}{n}$ where $\omega \to \infty$ and we randomly color the edges of $G_{n,p}$ with three colors, Red, Blue and Green. Show that w.h.p. there is a triangle in $G_{n,p}$ with every edge a different color.
Solution: Assume first that $np = \omega \leq \log n$ where $\omega = \omega(n) \to \infty$ and let Z be the number of multicolored triangles in $G_{n,p}$. Then

$$\mathbf{E}Z = \binom{n}{3} p^3 \times \frac{2}{9} \geq (1 - o(1))\frac{\omega^3}{27} \to \infty.$$

Next let $T_1, T_2, \ldots, T_M, M = \binom{n}{3}$ denote the triangles of K_n. Then if $T_i \in_m G_{n,p}$ means that T_i is in $G_{n,p}$ and is multicolored then

$$\begin{aligned}
\mathbf{E}Z^2 &= \sum_{i,j=1}^{M} \mathbf{P}(T_i, T_j \in_m G_{n,p}) \\
&= \sum_{i=1}^{M} \mathbf{P}(T_i \in_m G_{n,p}) \sum_{j=1}^{M} \mathbf{P}(T_j \in_m G_{n,p} \mid T_i \in_m G_{n,p}) \qquad (1) \\
&= M\mathbf{P}(T_1 \in_m G_{n,p}) \sum_{j=1}^{M} \mathbf{P}(T_j \in_m G_{n,p} \mid T_1 \in_m G_{n,p}) \qquad (2) \\
&= \mathbf{E}Z \times \sum_{j=1}^{M} \mathbf{P}(T_j \in_m G_{n,p} \mid T_1 \in_m G_{n,p}).
\end{aligned}$$

Here (2) follows from (1) by symmetry.

Now suppose that T_j, T_1 share σ_j edges. Then

$$\begin{aligned}
&\sum_{j=1}^{M} \mathbf{P}(T_j \in_m G_{n,p} \mid T_1 \in_m G_{n,p}) \\
&= 1 + \sum_{j:\sigma_j=1} \mathbf{P}(T_j \in_m G_{n,p} \mid T_1 \in_m G_{n,p}) + \\
&\qquad\qquad \sum_{j:\sigma_j=0} \mathbf{P}(T_j \in_m G_{n,p} \mid T_1 \in_m G_{n,p}) \\
&= 1 + 3(n-3)p^2 \times \frac{2}{9} + \left(\binom{n}{3} - 3n + 8\right) p^3 \times \frac{2}{9} \\
&\leq 1 + \frac{2\omega^2}{3n} + \mathbf{E}Z.
\end{aligned}$$

It follows that

$$\mathbf{Var}Z \leq (\mathbf{E}Z)\left(1 + \frac{2\omega^2}{3n} + \mathbf{E}Z\right) - (\mathbf{E}Z)^2 \leq 2\mathbf{E}Z.$$

Applying the Chebyshev inequality we get

$$\mathbf{P}(Z=0) \leq \mathbf{P}(|Z - \mathbf{E}Z| \geq \mathbf{E}Z) \leq \frac{\mathbf{Var}Z}{(\mathbf{E}Z)^2} \leq \frac{2}{\mathbf{E}Z} = o(1).$$

This proves the theorem for $p \leq \frac{\log n}{n}$. For larger p we can use monotonicity.