Homework 6: due November 11

1. Given $\mathcal{X}$ and an integer k we define the k-nearest neighbor graph $G_{k-NN,\mathcal{X}}$ as follows: We add an edge between x and y of $\mathcal{X}$ iff y is one of x's k nearest neighbors, in Euclidean distance or vice-versa. Show that if $k \geq C\log n$ for a sufficiently large C then $G_{k-NN,\mathcal{X}}$ is connected w.h.p.

   Solution: We know that if $\pi r^2 = (1+\varepsilon)\log n/n$ then $G_{\mathcal{X},r}$ is connected w.h.p. Now the Chernoff bounds imply that no ball of radius r round a point in $\mathcal{X}$ contains more than $10\log n$ points. So, w.h.p., $G_{k-NN,\mathcal{X}} \supseteq G_{\mathcal{X},r}$ if $C \geq 10$, and so is connected w.h.p.

2. A *tournament* T is an orientation of the complete graph K_n. In a random tournament, edge $\{u,v\}$ is oriented from u to v with probability 1/2 and from v to u with probability 1/2. Show that w.h.p. a random tournament is strongly connected.

   Solution: For each pair of vertices u,v, the probability that there is no w such that u,w,v is a path is $(3/4)^{n-2}$. So, the probability that a random tournament is not strongly connected is at most $n^2(3/4)^{n-2} = o(1)$.

3. Let T be a random tournament. Show that w.h.p. the size of the largest acyclic sub-tournament is asymptotic to $2\log_2 n$. (A tournament is acyclic if it contains no directed cycles).

   Solution: Let X_k denote the number of strongly connected acyclic sub-tournaments of size k. Then

$$\mathbb{E}(X_k) = \binom{n}{k} k! 2^{-k(k-1)/2} \leq (2^{-(k-1)/2} n)^k.$$

   So, $\mathbb{E}(X_k) \to 0$ if $k > 2\log_2 n + 2$ and this gives us an upper bound on the size of an acyclic sub-tournament. If $k = \lfloor (2-\varepsilon)\log_2 n \rfloor$ then $\mathbb{E}(X_k) \to \infty$ and we use the second moment method.

$$\begin{aligned}
&\mathbb{E}(X_k(X_k-1)) \\
&= \mathbb{E}(X_k) \sum_{\ell=0}^{k-1} \binom{k}{\ell} \binom{n-2k+\ell}{k-\ell} (k-\ell)! \left| \{(x_0,\ldots,x_{k-\ell} \geq 0) : x_0 + \cdots + x_{k-\ell} = k-\ell\} \right| 2^{-2\binom{k}{2}+\binom{\ell}{2}} \\
&= \mathbb{E}(X_k) \sum_{\ell=0}^{k-1} \binom{k}{\ell} \binom{n-2k+\ell}{k-\ell} (k-\ell)! \binom{2(k-\ell)}{k-\ell} 2^{-2\binom{k}{2}+\binom{\ell}{2}}. \qquad (1)
\end{aligned}$$

   Putting u_ℓ equal to the summand in (1) we see that

$$\frac{u_{\ell+1}}{u_\ell} = \frac{k-\ell}{\ell+1} \cdot \frac{1+O(k^2/n)}{n} \cdot \frac{(k-\ell-1)^2}{(2k-2\ell-1)(2k-2\ell)} \cdot 2^\ell \leq \frac{1}{\log n},$$

   assuming that $\varepsilon \geq 1/\log^{1/2} n$. So, $\mathbb{E}(X_k(X_k-1)) \sim \mathbb{E}(E_k)^2$ and we can use the Chebyshev inequality.