Homework 5: due October 21

1. Let $p = \frac{1000}{n}$ and $G = G_{n,p}$. Show that w.h.p. any red-blue coloring of the edges of G contains a mono-chromatic path of length $\frac{n}{1000}$. (Hint: Apply the argument of Section 6.3 of the book to both the red and blue sub-graphs of G to show that if there is no long monochromatic path then there is a pair of large sets S, T such that no edge joins S, T.)

   Solution: Running DFS on the graph G_R induced by the red edges, we see that if there is no red path of length $n/1000$ then we find sets D, U, A with $|D| = |U| \geq \frac{999n}{2000}$ such that there is no red edge between D and U. Similarly, $[n]$ can be partitioned into D', U', A' such that $|D'| = |U'| \geq \frac{999n}{2000}$ and there is no blue edge between D' and U'.

   Let $X = U \cap U', Y = U \cap D', X' = D \cap U', Y' = D \cap D'$ and let $x = |X|, y = |Y|, x' = |X'|, y' = |Y'|$. Then
   $$x + y = |U \cap (U' \cup D')| = |U \setminus A'| \geq \frac{999n}{2000} - \frac{n}{1000} = \frac{997n}{2000}. \tag{1}$$
   Similarly,
   $$x' + y', x + x', y + y' \geq \frac{997n}{2000}. \tag{2}$$
   It follows that either (i) $x, y' \geq \frac{997n}{4000}$ or (ii) $x', y \geq \frac{997n}{4000}$. (Failure of (i) and (ii) implies that (1) or (2) fail.) Suppose then that $x', y \geq \frac{997n}{2000}$. Now $X' \subseteq D$ and $Y \subseteq U$ and so there are no $X' : Y$ red edges. Furthermore, $X' \subseteq U'$ and $Y \subseteq D'$ and so there are no $X' : Y$ blue edges either. In other words $X' : Y = \emptyset$. But,
   $$\mathbb{P}\left(\exists \text{ disjoint } S, T : |S|, |T| \geq \frac{997n}{4000} \text{ and } S : T = \emptyset\right) \leq 2^{2n}\left(1 - \frac{1000}{n}\right)^{(997n/4000)^2} = o(1).$$

2. Show that w.h.p. the random 3-regular graph $G_{n,3}$ is not planar.

   Solution: The expected number X of edges in cycles of length at most $g = 100$ is
   $$\sum_{i=3}^{g} \binom{n}{i} \frac{(i-1)!}{2} \left(\frac{3}{3n/2 - 2g}\right)^i \leq 3^g.$$
   The Markov inequality implies that $X = O(\log n)$ w.h.p. Then after removing $O(\log n)$ edges we have a graph of girth at least g and at least $5n/4$ edges. This is not planar, since any planar n-vertex graph of girth g has at most $n(1 - 2/g)^{-1}$ edges.

3. Suppose that $1 \gg r \gg \sqrt{\frac{\log n}{n}}$. Show that w.h.p. the diameter of the random geometric graph $G_{\mathcal{X},r} = \Theta\left(\frac{1}{r}\right)$.

   Solution: W.h.p. there will be points in $A = [0, 1/4]^2$ and in $B = [3/4, 1]^2$. if $a \in A$ and $b \in B$ then $|a - b| \geq 1/2$. It follows that any path from a to b in $G = G_{\mathcal{X},r}$ has at least $1/4r$ edges.

   Conversely, partition $[0, 1]^2$ into cells of side $r/10$. W.h.p., each cell will contain points of $\mathcal{X}$. Also points in adjacent cells are adjacent in G. It follows that by going from adjacent cell to adjacent cell we can reach b from a in at most $20/r$ steps.