Homework 1: due September 4Solutions

In all of these questions, the graph in question is $G_{n,p}$.

1. Suppose that $p = \frac{\log n}{n}$. Let $S = \left\{v : degree(v) \leq \frac{\log n}{20}\right\}$. Prove that w.h.p. S contains no edges.

   Solution: Let X denote the number of pairs $v, w \in [n]$ such that (i) there is an edge $\{v, w\}$ in $G_{n,p}$ and $degree(v), degree(w) \leq \log n/20$. Then $X = 0$ implies that S contains no edges.

$$\mathbb{E}(X) \leq \binom{n}{2} p \left(\sum_{k=0}^{\log n/20} \binom{n-2}{k} p^k (1-p)^{n-k} \right)^2$$
$$\leq n \log n \left(\sum_{k=0}^{\log n/20} u_k \right)^2,$$

   where $u_k = \binom{n}{k} p^k (1-p)^{n-k}$. Now if $k \leq \log n/20$ then

$$\frac{u_k}{u_{k-1}} = \frac{(n-k)p}{(k+1)(1-p)} \geq \frac{\left(1 - O\left(\frac{\log n}{n}\right)\right)}{\left(1 - O\left(\frac{1}{\log n}\right)\right) \frac{\log n}{20}} \geq 19.$$

   So,

$$\sum_{k=0}^{\log n/20} u_k \leq u_{\log n/20} \sum_{r=0}^{\infty} \frac{1}{(19)^r} = \frac{19 u_{\log n/20}}{18}.$$

   Now

$$u_{\log n/20} \leq \left(\frac{nep}{\log n/20} \right)^{\log n/20} e^{-(n - \log n/20)p} \leq (1 + o(1))(20e)^{\log n/20} n^{-1} \leq n^{-4/5}.$$

   So,

$$\mathbb{E}(X) \leq n \log n \times \frac{19}{18} \times n^{-8/5} = o(1).$$

2. Suppose that $p = \frac{c}{n}$ where c is a constant and that $s_0 = n/(e^2c^2)$. Show that w.h.p. all sets of vertices of size $s \leq s_0$ contain at most $2s$ edges.

Solution: Let X de note the number of sets S of size at most s_0 that contain $2|S|$ edges. Then

$$\begin{aligned}
\mathbb{E}(X) &\leq \sum_{s=4}^{s_0} \binom{n}{s} \mathbb{P}([s] \text{ contains } 2s \text{ edges}) \\
&\leq \sum_{s=4}^{s_0} \left(\frac{ne}{s}\right)^s \mathbb{E}(\text{number of sets of } 2s \text{ edges in } [s] \\
&\leq \sum_{s=4}^{s_0} \left(\frac{ne}{s}\right)^s \binom{\binom{s}{2}}{2s} \left(\frac{c}{n}\right)^{2s} \\
&\leq \sum_{s=4}^{s_0} \left(\frac{ne}{s}\right)^s \left(\frac{es^2c}{2ns}\right)^{2s} \\
&\leq \sum_{s=4}^{s_0} \left(\frac{e^3c^2s}{4n}\right)^s .
\end{aligned}$$

Let $u_s = \left(\frac{e^3c^2s}{4n}\right)^s$. If $s \leq \log^2 n$ then $u_s \leq n^{-1/2}$ and so $\sum_{s=4}^{\log^2 n} u_s \leq \frac{\log n}{n^{1/2}} = o(1)$. If $s > \log^2 n$ then $u_s \leq (e/4)^{\log^2 n}$ and so $\sum_{\log^2 n}^{s_0} u_s \leq n(e/4)^{\log^2 n} = o(1)$.

3. Suppose that $p = \frac{c}{n}$ where c is a constant. Show that w.h.p. there are no two cycles of size at most 10 that share a vertex.

   Solution: If S is the set of vertices of two cycles thaty share a vertex then S has at most $s \leq 19$ vertices and contains at least $s+1$ edges. So

$$\mathbb{P}(\exists\ cycles) \leq \sum_{s=4}^{19} \binom{n}{s} \binom{\binom{s}{2}}{s+1} \left(\frac{c}{n}\right)^{s+1} = O\left(\frac{1}{n}\right) = o(1).$$