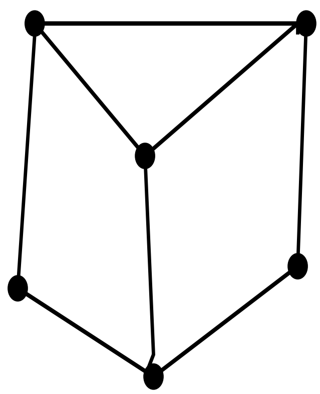
Planar Graphs

A graph $G = (V, E)$ is *planar* if it can be “drawn” on the plane without edges crossing except at endpoints – a *planar embedding* or *plane graph*.

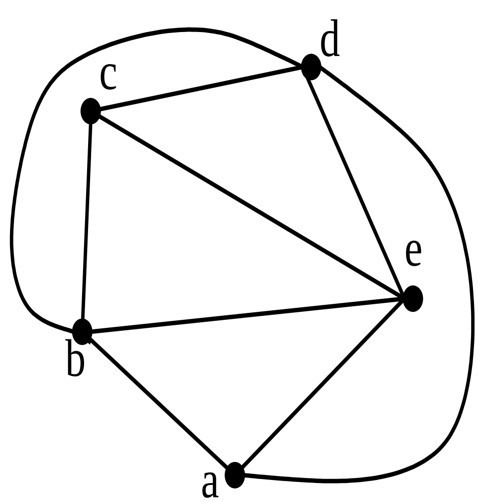
More precisely: there is a 1-1 function $f : V \to \boldsymbol{R}^2$ and for each $e \in E$ there exists a 1-1 continuous $g_e : [0, 1] \to \boldsymbol{R}^2$ such that

(a) $e = xy$ implies $f(x) = g_e(0)$ and $f(y) = g_e(1)$.
(b) $e \neq e'$ implies that $g_e(x) \neq g_{e'}(x')$
for all $x, x' \in (0, 1)$.

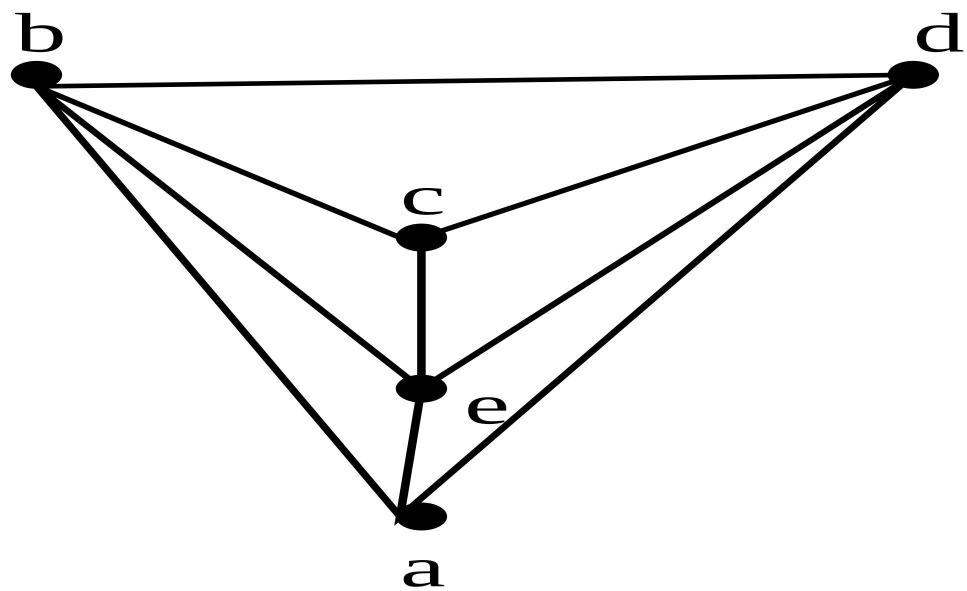
g_e or its image is referred to as a *curve*.



Planar



- **Theorem (Fáry)**
  A simple planar graph has an embedding in which all edges are straight lines.

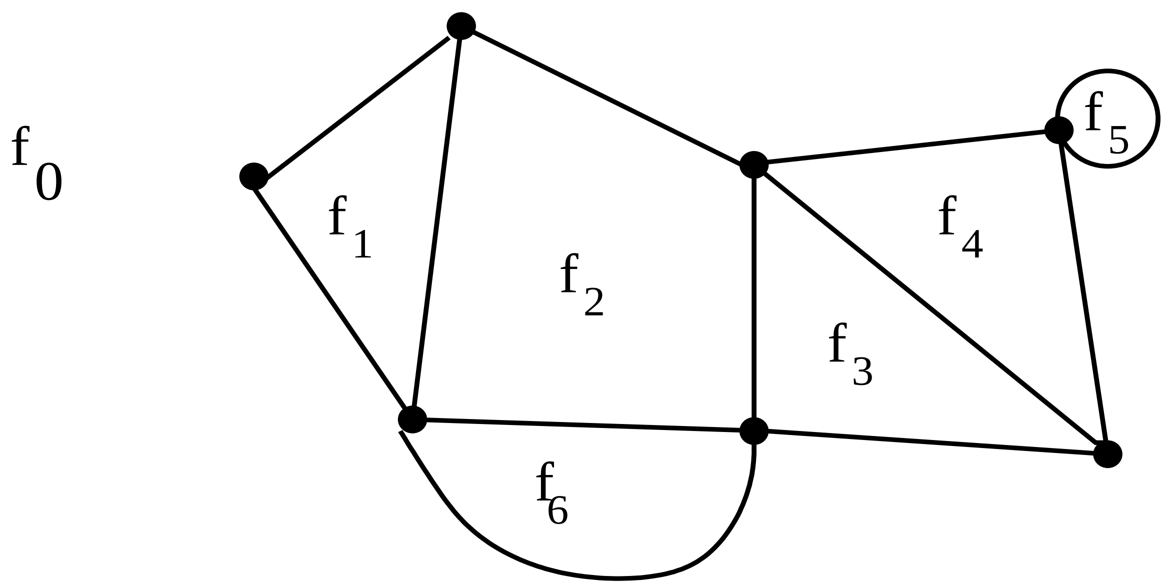


- Not all graphs are planar.

- Graphs can have several non-isomorphic embeddings.

Faces

Given a plane graph G, a face is a maximal region S such that $x, y \in S$ implies that x, y can be joined by a curve which does not meet any edge of the embedding.



The above embedding has 7 faces.
f_0 is the *outer* or *infinite* face.

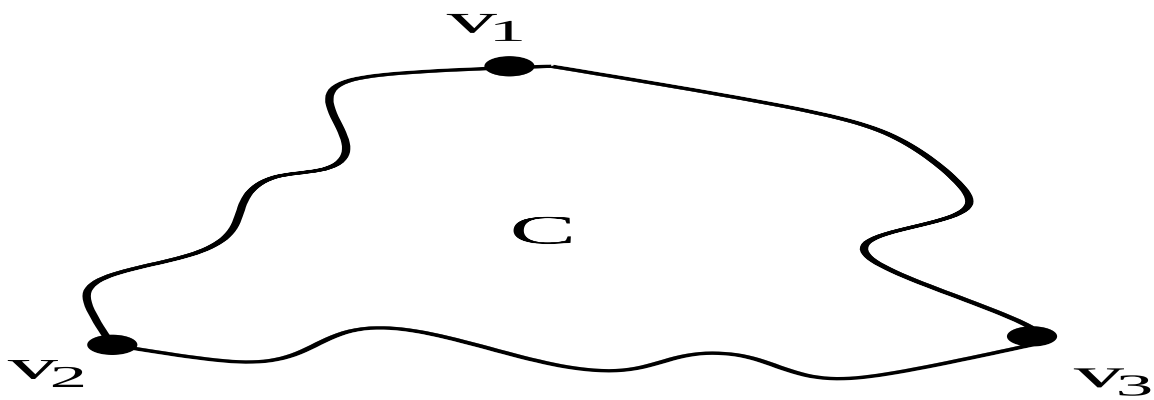
$\phi(G)$ is the number of faces of G.

Jordan Curve Theorem

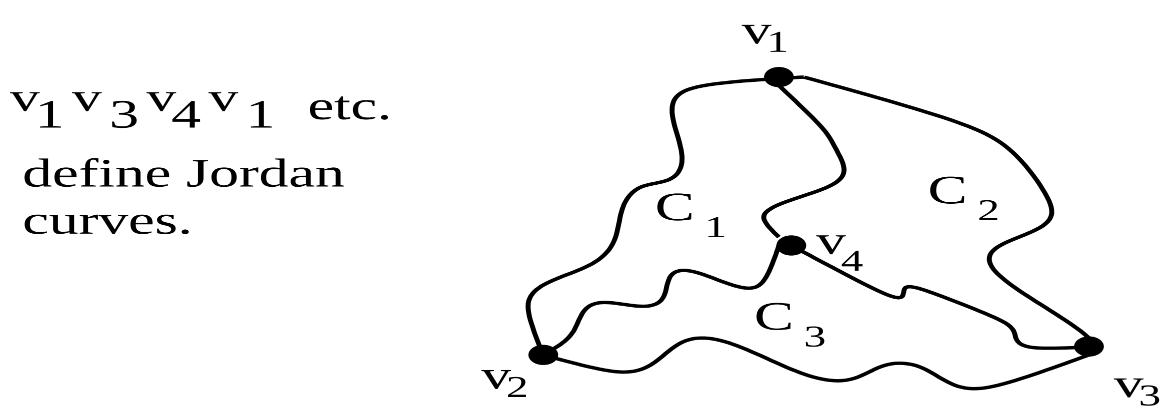
If f is a 1-1 continuous map from the circle $S^1 \to \boldsymbol{R}^2$ then f partitions $\boldsymbol{R}^2 \setminus f(S^1)$ into two disjoint connected open sets $Int(f)$, $Ext(f)$. The former is bounded and the latter is unbounded.

As a consequence, if $x \in Int(f)$, $y \in Ext(f)$ and x, y are joined by a closed curve C in $\boldsymbol{R}^2$ then C meets $f(S^1)$.

K_5 is not planar.



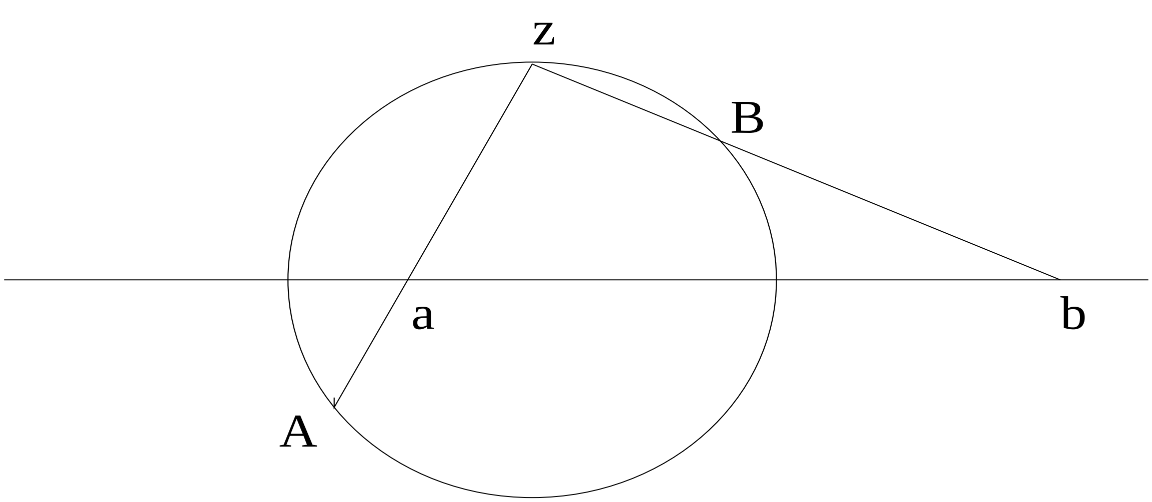
v_4 is inside or outside of C – assume inside.



Now no place to put v_5 – e.g. if we place v_5 into C_1 then the v_5v_3 curve crosses the boundary of C_1.

Stereographic Projection

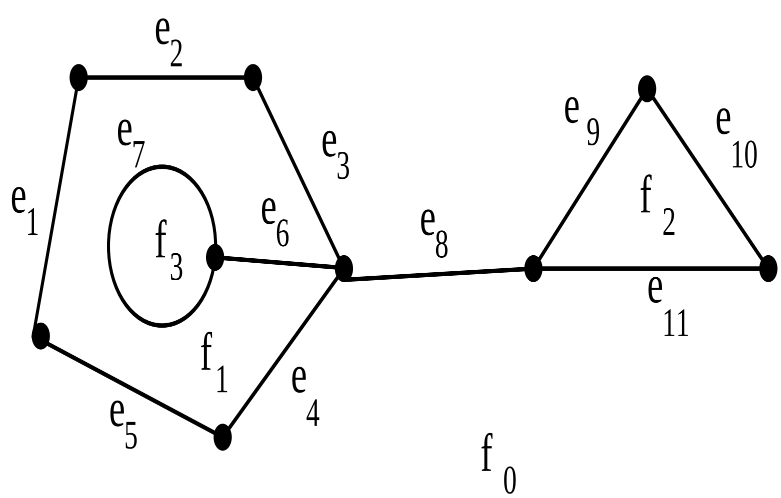
A graph is embeddable in the plane iff it is embeddable on the surface of a sphere.



$f : \boldsymbol{R}^2 \to S^2 \setminus \{z\}$. $f(x, y) = \left(\frac{2x}{\rho}, \frac{2y}{\rho}, \frac{\rho-2}{\rho}\right)$ where $\rho = 1 + x^2 + y^2$.

Given an embedding on the sphere we can choose z to be any point not an edge or vertex of the embedding. Thus if v is a vertex of a plane graph, G can be embedded in the plane so that v is on the outer face.

The *boundary* $b(f)$ of face f of plane graph G is a closed clockwise walk around the edges of the face.



$$\begin{aligned}
b(f_0) &= e_1e_2e_3e_8e_9e_{10}e_{11}e_8e_4e_5 \\
b(f_1) &= e_1e_2e_3e_6e_7e_6e_4e_5 \\
b(f_2) &= e_9e_{10}e_{11} \\
b(f_3) &= e_7
\end{aligned}$$

The degree $d(f)$ of face f is the number of edges in $b(f)$.

Each edge appears twice as an edge of a boundary and so if F is the set of faces of G, then

$$\sum_{f \in F} d(f) = 2\epsilon.$$

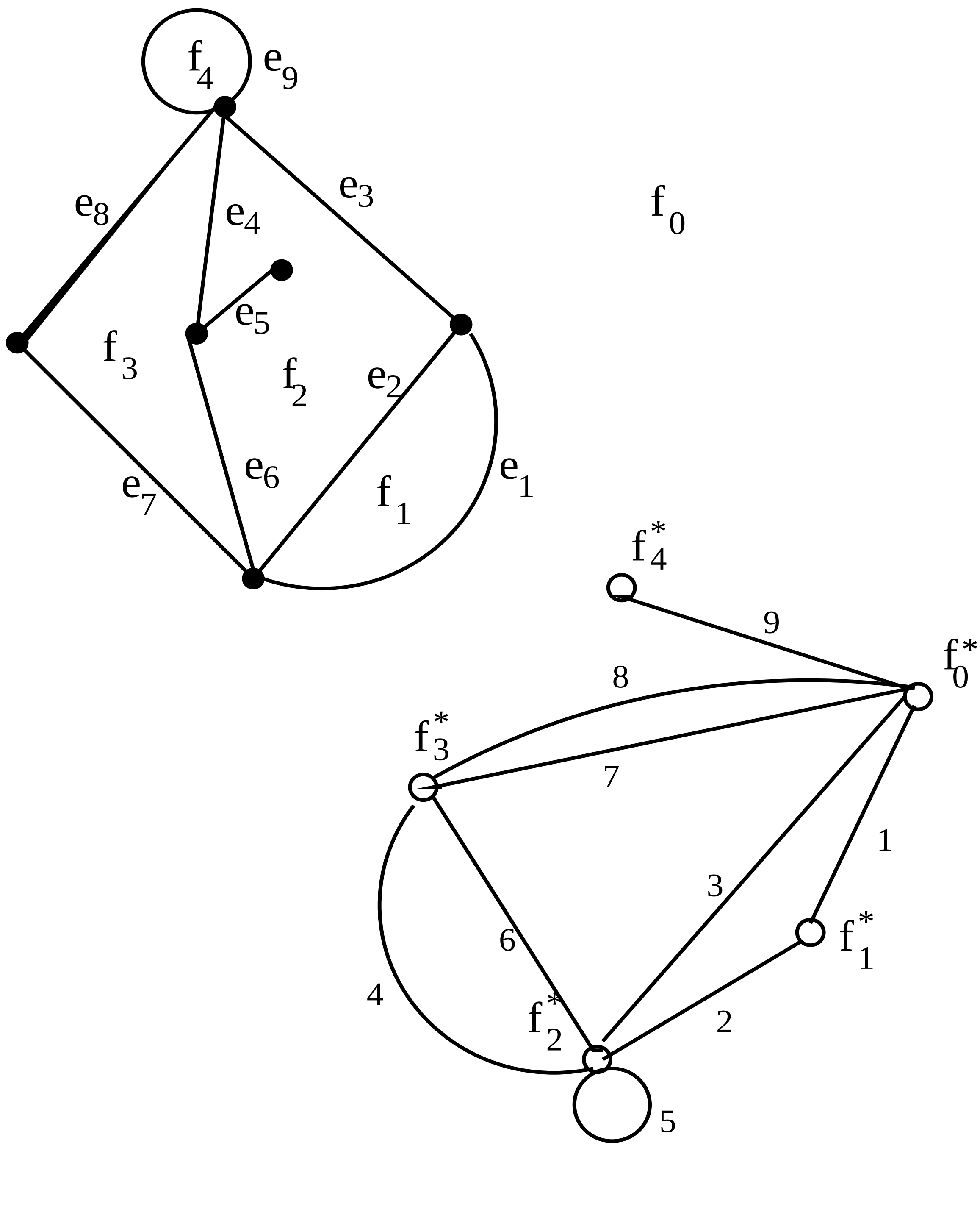
A cut edge like e_6 appears twice in the boundary of a single face.

Dual Graphs

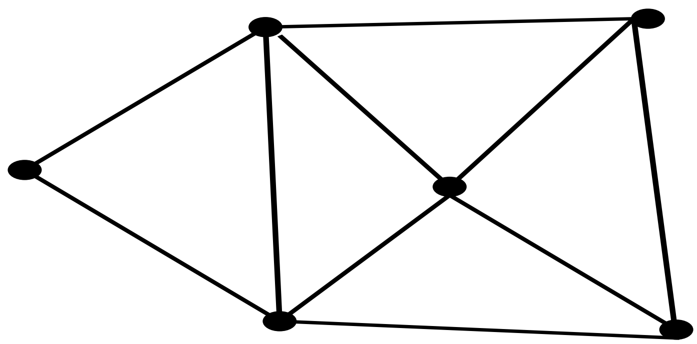
Let G be a plane graph. We define its dual $G^* = (V^*, E^*)$ as follows: There is a vertex f^* corresponding to each face f of G.
There is an edge e^* corresponding to each edge e of G.
f^* and g^* are joined by edge e^* iff edge e is on the boundary of f and g.
Cut edges yield loops.

Theorem 1
(a) G^* is planar.
(b) G connected implies $G^{**} = G$.

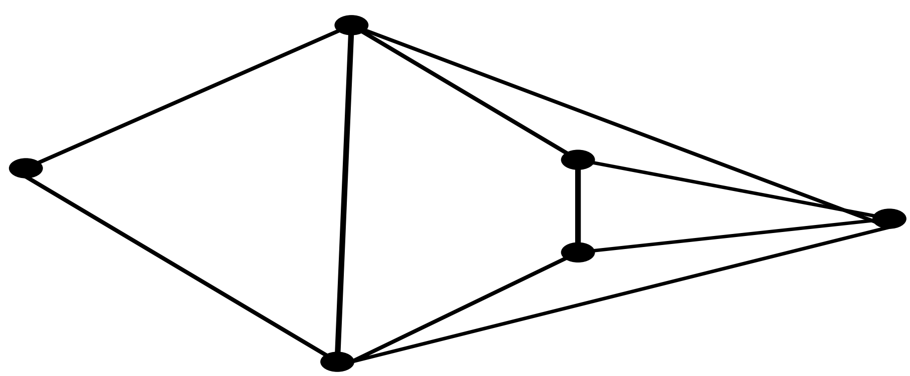
$\square$


f_4
e_9
e_8
e_4
e_3
f_0
e_5
f_3
f_2
e_2
e_6
e_7
f_1
e_1
f_4^*
9
f_0^*
8
f_3^*
7
1
3
f_1^*
6
4
f_2^*
2
5

The following is possible: start with planar graph G and form 2 distinct embeddings G_1, G_2. The duals G_1^*, G_2^* may not be isomorphic.



$\mathbf{G_1}$



$\mathbf{G_2}$

G_1 has a face of degree 5 and so G_1^* has a vertex of degree 5. G_2^* has maximum degree 4.

Thus duality is a meaningfull notion w.r.t. plane graphs and not planar graphs.

$\phi(G)$ is the number of faces of plane graph G.

(a) $\nu(G^*) = \phi(G)$.

(b) $\epsilon(G^*) = \epsilon(G)$.

(c) $d_{G^*}(f^*) = d_G(f)$.

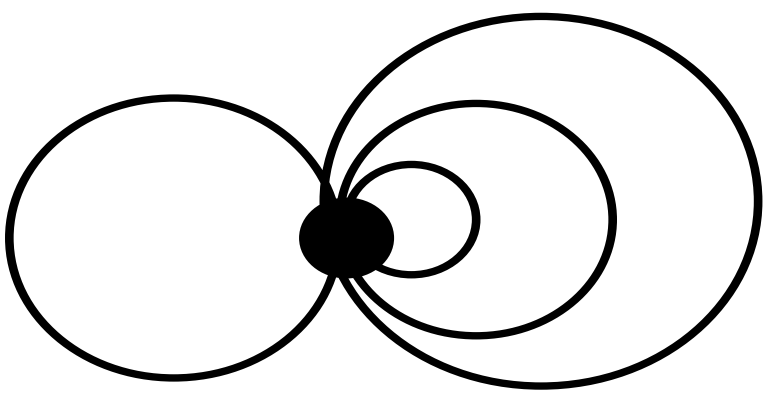
Note that (c) says that the *degree* of f^* in G^* is equal to the size of the boundary of f in G.

Euler's Formula

Theorem 2 *Let G be a connected plane graph. Then*

$$\nu - \epsilon + \phi = 2.$$

Proof By induction on ν.
If $\nu = 1$ then G is a collection of loops.



$$\phi = \epsilon + 1.$$

If $\nu > 1$ there must be an edge e which is not a loop.
Contract e to get $G \cdot e$.
$G \cdot e$ is connected.

$$\begin{aligned}
\phi(G \cdot e) &= \phi(G) \\
\nu(G \cdot e) &= \nu(G) - 1 \\
\epsilon(G \cdot e) &= \epsilon(G) - 1
\end{aligned}$$

But then

$$\begin{aligned}
\nu(G) - \phi(G) + \epsilon(G) &= \nu(G \cdot e) - \phi(G \cdot e) + \epsilon(G \cdot e) \\
&= 2
\end{aligned}$$

by induction. $\square$

Corollary 1 *All plane embeddings of a planar graph* G *have the same number* $\epsilon - \nu + 2$ *faces.*

Corollary 2 *If* G *is a simple plane graph with* $\nu \geq 3$ *then*

$$\epsilon \leq 3\nu - 6.$$

Proof Every face has at least 3 edges. Thus

$$2\epsilon = \sum_{f \in F} d(f) \geq 3\phi. \tag{1}$$

Thus by Euler's formula,

$$\nu - \epsilon + \frac{2}{3}\epsilon \geq 2.$$

□

It follows from the above proof that if $\epsilon = 3\nu - 6$ then there is equality in (1) and so every face of G is a triangle.

Corollary 3 *If G is a planar graph then $\delta(G) \leq 5$.*

Proof

$$\nu\delta \leq 2\epsilon \leq 6\nu - 12.$$

$\square$

Corollary 4 *If G is a planar graph then $\chi(G) \leq 6$.*

Proof Since each subgraph H of G is planar we see that the colouring number $\delta^*(G) \leq 5$. $\square$

Corollary 5 *K_5 is non-planar.*

Proof

$$\epsilon(K_5) = 10 > 3\nu(K_5) - 6 = 9.$$

$\square$

Corollary 6 $K_{3,3}$ *is non-planar.*

Proof $K_{3,3}$ has no odd cycles and so if it could be embedded in the plane, every face would be of size at least 4. In which case

$$4\phi \leq \sum_{f \in F} d(f) = 2\epsilon = 18$$

and so $\phi \leq 4$.

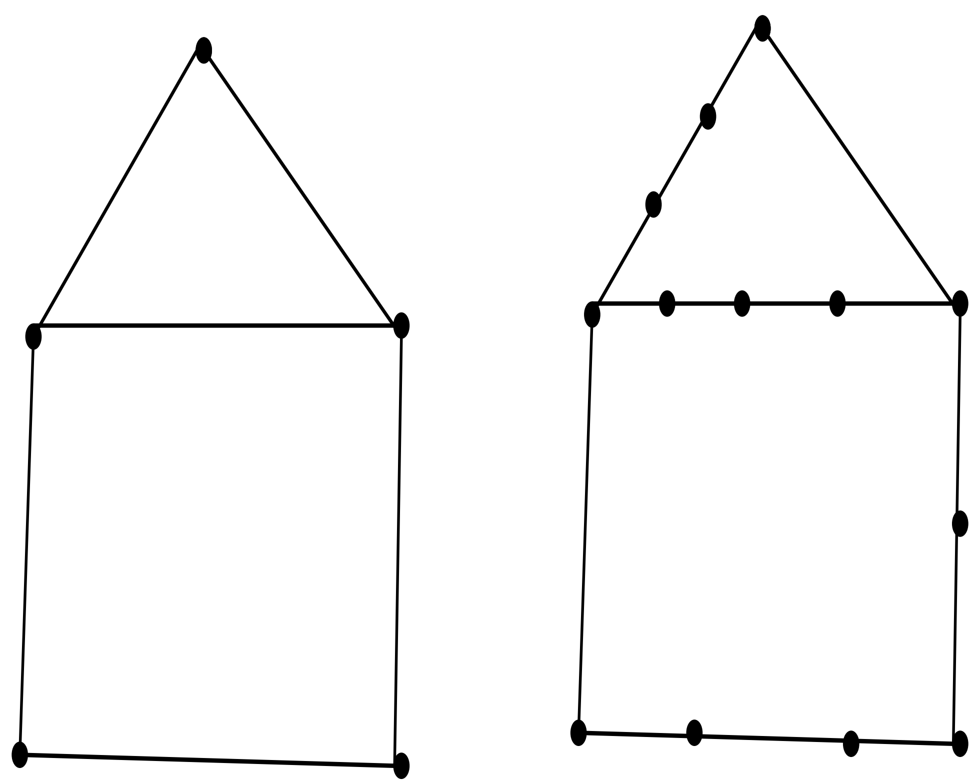
But then from Euler's formula,

$$2 = 6 - 9 + \phi \leq 1,$$

contradiction. $\square$

Kuratowski's Theorem

A *sub-division* of a graph G is one which is obtained by replacing edges by (vertex disjoint) paths.



Clearly, if G is planar then any sub-division of G is also planar.

Theorem 3 *A graph is non-planar iff it contains a sub-division of $K_{3,3}$ or K_5.*

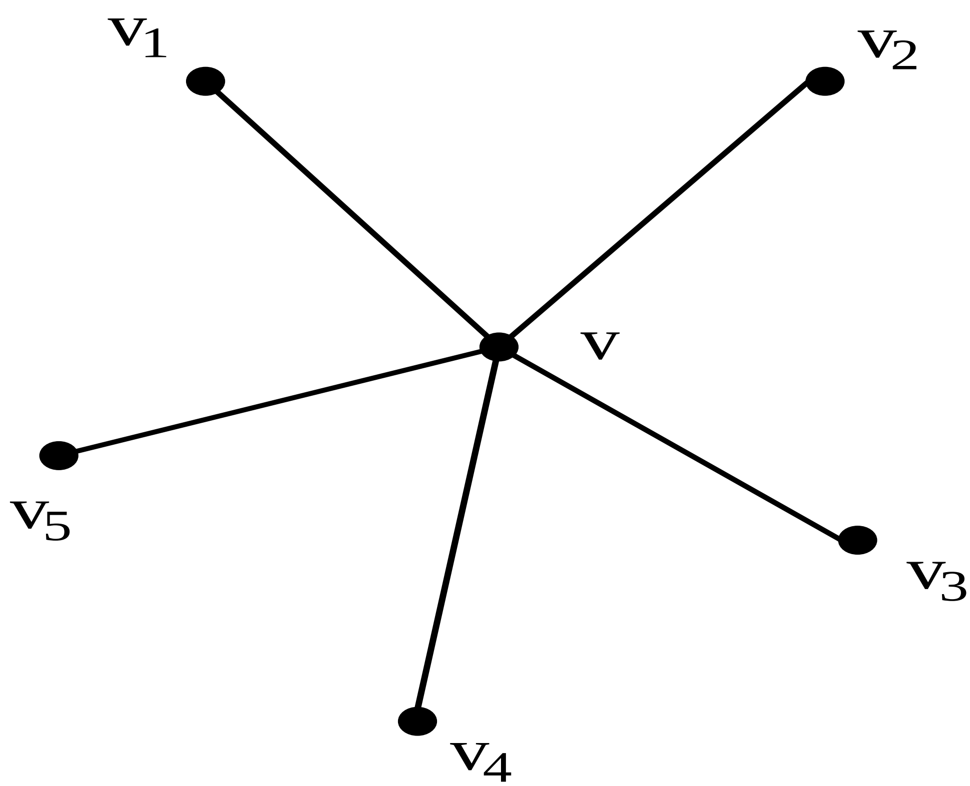
Theorem 4 *If G is planar then $\chi(G) \leq 5$.*

By induction on ν. Trivial for $\nu = 1$.

Suppose G has $\nu > 1$ vertices and the result is true for all graphs with fewer vertices. G has a vertex v of degree at most 5. $H = G - v$ can be properly 5-coloured, induction.

If $d_G(v) \leq 4$ then we can colour v with a colour not used by one of its neighbours.

Suppose $d_G(v) = 5$. Take some planar embedding.

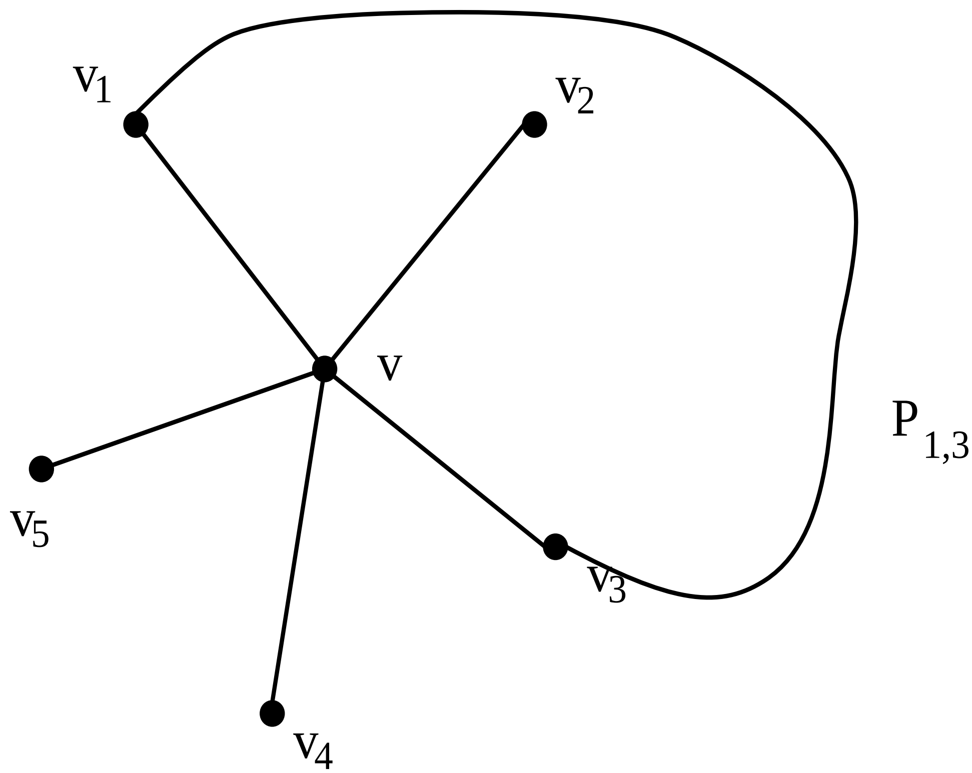


$H = G - v$ can be 5-coloured. We can assume that $c(v_i) \neq c(v_j)$ for $i \neq j$ else we can extend the colouring c to v as previously. We can also assume that $c(v_i) = i$ for $1 \leq i \leq 5$.

Let $K_i = \{u \in V - v : c(u) = i$ for $1 \leq i \leq 5$ and let $H_{i,j} = H[K_i \cup K_j]$ for $1 \leq i < j \leq 5$.

First consider $H_{1,3}$. If v_1 and v_3 belong to different components C_1, C_3 of $H_{1,3}$ then we can interchange the colours 1 and 3 in C_1 to get a new proper colouring c' of H with $c'(v_1) = c'(v_3) = 3$ which can then be extended to v.

So assume that there is a path $P_{1,3}$ from v_1 to v_3 which only uses vertices from $K_1 \cup K_3$. Assume w.l.o.g. that v_2 is inside the cycle $(v_1, v, v_3, P_{1,3}, v_1)$,



Now consider $H_{2,4}$. We claim that v_2 and v_4 are in different components C_2, C_4, in which case we can interchange the colours 2 and 4 in C_2 to get a new colouring c'' with $c''(v_2) = c''(v_4)$.

If v_2 and v_4 are in the same component of $H_{2,4}$ then there is a path $P_{2,4}$ from v_2 to v_4 which only uses vertices of colour 2 or 4. But this path would have to cross $P_{1,3}$ which only uses vertices of colour 1 and 3 – contradiction. $\square$